

# $S_4$ flavor symmetry and fermion masses: towards a grand unified theory of flavor

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ABSTRACT: Pursuing a bottom-up approach to explore which flavor symmetry could serve as an explanation of the observed fermion masses and mixings, we discuss an extension of the standard model (SM) where the flavor structure for both quarks and leptons is determined by a spontaneously broken  $S_4$  and the requirement that its particle content is embeddable simultaneously into the conventional  $SO(10)$  grand unified theory (GUT) and a continuous flavor symmetry  $G_f$  like  $SO(3)_f$  or  $SU(3)_f$ . We explicitly provide the Yukawa and the Higgs sector of the model and show its viability in two numerical examples which arise as small deviations from rank one matrices. In the first case, the corresponding mass matrix is democratic and in the second one only its 2-3 block is non-vanishing. We demonstrate that the Higgs potential allows for the appropriate vacuum expectation value (VEV) configurations in both cases, if CP is conserved. For the first case, the chosen Yukawa couplings can be made natural by invoking an auxiliary  $Z_2$  symmetry. The numerical study we perform shows that the best-fit values for the lepton mixing angles  $\theta_{12}$  and  $\theta_{23}$  can be accommodated for normal neutrino mass hierarchy. The results for the quark mixing angles turn out to be too small. Furthermore the CP-violating phase  $\delta$  can only be reproduced correctly in one of the examples. The small mixing angle values are likely to be brought into the experimentally allowed ranges by including radiative corrections. Interestingly, due to the  $S_4$  symmetry the mass matrix of the right-handed neutrinos is proportional to the unit matrix.

KEYWORDS: Quark Masses and SM Parameters, Neutrino Physics, Discrete and Finite Symmetries, GUT.

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## 1. Introduction

The discovery of neutrino masses and attempts to understand the flavor puzzle have made it quite natural to expect the existence of an embedding product group for the SM such as  $SO(10) \times G_f$  [1, 2] at very high energies, where  $SO(10)$  acts as gauge and  $G_f$  as a flavor (family) symmetry. All fermions of one generation can then be unified into the spinor representation **16** of  $SO(10)$  and the three known generations are assigned to representations of  $G_f$ . To specify  $G_f$  two important properties have to be fixed:  $G_f$  can be either abelian or non-abelian and it can be either continuous or discrete. Abelian symmetries are

not able to explain the existence of more than one generation, since their representations are all one-dimensional. Therefore we consider such a choice of  $G_f$  to be less interesting than the others, although many models have obtained interesting results from U(1) flavor symmetries [3]. Most of the other models in which  $G_f$  is non-abelian [1] have some common features, like additional U(1) or  $Z_n$  factors, heavy vector-like fermions, elaborate mechanisms for the VEV alignment, to make them viable. Further unification might explain these assumptions. We search for a simpler model by first constructing a low energy theory with the SM gauge group and a discrete non-abelian  $G_f$  and then showing possible embeddings of this theory into an SO(10) GUT and a continuous  $G_f$ . To really unify all three generations  $G_f$  should be either  $SO(3)_f$  or  $SU(3)_f$ . Our discrete symmetry will therefore be a subgroup of SO(3) or SU(3). We like to point out that none of the models mentioned in reference [1] with  $G_f$  being continuous follows this strategy. In cases where  $G_f$  is discrete [2], the symmetry is always broken at the GUT scale so that these models are also not comparable to our ansatz. Models in which the discrete non-abelian flavor symmetry is only broken at low energies became very popular in the last few years [4–11]. However, these models can rarely be embedded into  $SO(10) \times G_f$ .

In working with discrete symmetries two issues are necessary to mention. First, the breaking of a discrete global symmetry does not lead to (unwanted) massless Goldstone bosons, unlike continuous symmetries. Second, if this breaking is only spontaneous, it might produce domain walls which can be a serious problem [12]. It can be solved by either invoking low scale inflation or embedding the discrete symmetry into a continuous gauge symmetry [13]. A further issue which is only important, if the symmetry should be gauged, is the question of anomalies. Since we intend to embed the discrete symmetry into a continuous one at high energies, it is enough to make sure that the continuous one is anomaly-free. This can be checked by calculating the usual triangle diagrams. If the symmetry turns out to be anomalous, adding appropriate representations can solve this problem.

Searching for an adequate discrete group we concentrate on the smallest subgroups of SO(3) or SU(3) which have at least one irreducible three-dimensional representation.  $A_4$  is the smallest of these groups and has already been discussed extensively in the literature [5]. The second smallest group, sometimes called  $T_7$ , is rarely known, but its properties are in some sense similar to  $A_4$ . Concerning the three-dimensional representations the same holds for the groups  $T'$  [10] ( $T'$  two-valued group of  $T$  and  $T$  isomorphic to  $A_4$ ) and  $T_h$  ( $T_h$  isomorphic to  $A_4 \times C_2$ ). In this paper, we focus on the group  $S_4$ , which has a different group structure and is therefore worth exploring as a possible flavor symmetry. This has already been done in several papers [6, 7]. Pakvasa et al. [6] constructed an SM-like model for quarks whose transformation properties under  $S_4$ , however, do neither allow an embedding into SO(10) nor into  $SO(3)_f$  ( $SU(3)_f$ ). An SO(10) model presented by one of the authors (R.N.M.) and Lee [7] uses similar  $S_4$  representations for the fermions and Higgs fields, but leads to the SMA solution to the solar neutrino problem which is ruled out <sup>1</sup>. Recently,

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<sup>1</sup>The renormalization group extrapolations of neutrino masses in this model have not been studied and they might make a difference to the above conclusion since neutrinos are quasi-degenerate in this case.

also Ma [6] discussed an  $S_4$  model, but he only considered the neutrino sector. Therefore there is still no working  $S_4$  model which addresses the flavor structure of all fermions in a unified manner and can be embedded into a GUT as well as a continuous flavor symmetry. The aim of our study is to fill this gap.

As noted, in this paper, we consider the non-abelian discrete flavor symmetry  $S_4$  accompanying the SM gauge group which naturally commutes with it. The gauge and the flavor symmetry are assumed to be both broken only spontaneously at the electroweak scale. The particle content consists of the three known fermion generations together with three right-handed neutrinos and a set of six Higgs fields being SM doublets and transforming as  $\underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$  under  $S_4$ . The choice of the fermion content allows the embedding of our model into  $\text{SO}(10) \times G_f$ . We observe that the number of parameters determining the mass matrices in our model, i.e. the Yukawa couplings and the VEVs of the Higgs fields, equals the number of observables, i.e. masses and mixing angles, in the CP-conserving case. If CP is violated, there are more parameters in the model than observables to fit. However, it is rather non-trivial to find parameter configurations which can reproduce all data, since  $S_4$  constrains the mass structures as well as the Higgs potential. In our numerical study this turns out to be possible except for rather small deviations of the quark mixings which might be fixed by radiative corrections.

The paper is organized as follows: section 2 contains the group theory of  $S_4$ , in section 3 we present our model and argue why this is the minimal model with respect to the possible embeddings into  $\text{SO}(10)$  and  $G_f$ . Then, we display the structure of the mass matrices arising from the  $S_4$  invariant Yukawa couplings. After setting our conventions for the mixing matrices we give some numerical examples and comment on their viability. To complete the model, we calculate the Higgs potential and perform a restricted analysis of its possible minima. In section 6 we conclude and point to open questions in our model. Finally, the appendices contain Kronecker products, Clebsch Gordan coefficients and embeddings of  $S_4$  as well as some information on the Higgs potential and the selected minima.

## 2. The $S_4$ group

The group  $S_4$  is the permutation group of four distinct objects. It is isomorphic to the group  $O$  which is the symmetry group of a regular octahedron and so well-known in solid state physics. Its order is 24, i.e. it has 24 distinct elements.  $S_4$  has five conjugate classes and therefore contains five irreducible representations which are all real. Among these are two one-dimensional ones, the identity (i.e. the representation being invariant under all transformations of  $S_4$ , also called the symmetric representation) and the anti-symmetric one (i.e. the one changing sign under odd permutations, also called alternating). In the following we will denote the identity one with  $\underline{\mathbf{1}}_1$  and the anti-symmetric one with  $\underline{\mathbf{1}}_2$ . There is one two-dimensional representation called  $\underline{\mathbf{2}}$  and two three-dimensional ones,  $\underline{\mathbf{3}}_1$  and  $\underline{\mathbf{3}}_2$ . Out of these five irreducible representations only the two three-dimensional ones are faithful. Their characters  $\chi$ , i.e. the traces of their representation matrices, are given in the character table, see table 1. There we use the following notations:  $\mathcal{C}_i$  with  $i = 1, \dots, 5$

	classes				
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
G	$\mathbb{1}$	$A^2$	$AB^2$	B	A
${}^\circ\mathcal{C}_i$	1	3	6	8	6
${}^\circ\text{h } \mathcal{C}_i$	1	2	2	3	4
$\underline{\mathbf{1}}_1$	1	1	1	1	1
$\underline{\mathbf{1}}_2$	1	1	-1	1	-1
$\underline{\mathbf{2}}$	2	2	0	-1	0
$\underline{\mathbf{3}}_1$	3	-1	1	0	-1
$\underline{\mathbf{3}}_2$	3	-1	-1	0	1

**Table 1:** Character table of the group  $S_4$ . (explanation see text)

are the five classes of the group,  ${}^\circ\mathcal{C}_i$  is the order of the  $i^{\text{th}}$  class, i.e. the number of distinct elements contained in this class,  ${}^\circ\text{h } \mathcal{C}_i$  is the order of the elements  $R$  in the class  $\mathcal{C}_i$ , i.e. the smallest integer ( $> 0$ ) for which the equation  $R^{{}^\circ\text{h } \mathcal{C}_i} = \mathbb{1}$  holds. Furthermore the table contains one representative for each class  $\mathcal{C}_i$  given as product of the generators A and B of the group. From the generators A and B all other elements of  $S_4$  can be formed by multiplication. They ought to fulfill the following relations [14]:

$$A^4 = \mathbb{1}, \quad B^3 = \mathbb{1} \quad \text{and} \quad AB^2A = B, \quad ABA = BA^2B. \quad (2.1)$$

We show one possible choice of generators in appendix A.1. Using them we calculate the Clebsch Gordan coefficients for all the Kronecker products.

$S_4$  is the smallest group containing one-, two- and three-dimensional representations together with the group  $T'$ .

$S_4$  can be embedded into  $SO(3)$  as well as in  $SU(3)$  (where it is isomorphic to the group  $\Delta(24)$  [15]) and therefore gives the opportunity to embed our discrete flavor symmetry into a continuous one which is broken at a high energy scale. Possible embedding schemes are shown in appendix A.4.

In our model the group  $S_4$  is broken completely at the electroweak scale, however this breaking could also occur in two steps such that  $S_4$  breaks to one of its subgroups which is then completely broken. The non-abelian subgroups of  $S_4$  turn out to be already well-known as flavor symmetries: they are  $S_3$  (which is isomorphic to  $D_3$ ),  $D_4$  and  $A_4$ . Correlation tables containing the corresponding breaking sequences for the representations of  $S_4$  can be found in [16].

### 3. The $S_4$ model

#### 3.1 Particle assignment

In this subsection, we describe how to assign the fermions and Higgs bosons to different  $S_4$  representations in such a way that the model can be embedded simultaneously into  $SO(10)$

and a continuous flavor symmetry  $G_f$ . We argue that our choice which is displayed in table 2 is unique in that sense.

To embed the model into  $SO(10)$  all the fermion generations have to transform in the same way under  $S_4$ . Furthermore they have to transform either as trivial or as fundamental representation of the flavor group  $G_f$ , since all the other representations of these groups have a dimension larger than three. The only choice is then that they transform as  $\underline{\mathbf{3}}_2$  under  $S_4$  apart from the trivial one where all the generations just form total singlets under the flavor group. An important point to note is that it is not possible to assign the three generations to  $\underline{\mathbf{3}}_1$ , since this representation cannot be identified with the fundamental representation of  $SO(3)_f$  or  $SU(3)_f$ . Therefore our assignment is unique.

In the next step, we have to choose the representations for the Higgs fields in order to give masses to the SM fermions. This choice depends on the desired Yukawa structure as well as the constraint to fill (a) certain representation(s) of the embedding group  $G_f$ . For fermions transforming as  $\underline{\mathbf{3}}_2$  under  $S_4$ , the Higgs fields which can couple in an  $S_4$  invariant manner belong either to  $\underline{\mathbf{1}}_1$ ,  $\underline{\mathbf{2}}$ ,  $\underline{\mathbf{3}}_1$  or to  $\underline{\mathbf{3}}_2$ , i.e. the only representation which cannot couple to form a total singlet under  $S_4$  is  $\underline{\mathbf{1}}_2$ . Taking now for simplicity only the couplings which are symmetric in flavor space, i.e. lead to symmetric mass matrices, one is left with the representations  $\underline{\mathbf{1}}_1$ ,  $\underline{\mathbf{2}}$  and  $\underline{\mathbf{3}}_1$ . Regarding the possible embeddings into  $SO(3)_f$  one recognizes that at least five Higgs fields transforming as  $\underline{\mathbf{2}} + \underline{\mathbf{3}}_1$  are needed and for an embedding into  $SU(3)_f$  one needs six fields  $\sim \underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$ . Furthermore it turns out that the minimal version of five Higgs fields  $\sim \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$  is not phenomenologically viable, since it leads to traceless symmetric mass matrices. Therefore the minimal setup of Higgs fields that we choose to get fermion masses contains six fields transforming as  $\underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$  under  $S_4$ . In the case of an embedding into  $SO(3)_f$  these representations are identified with  $\underline{\mathbf{1}} + \underline{\mathbf{5}}$  and in the case of  $SU(3)_f$  they are unified into the six-dimensional representation of  $SU(3)_f$  (see: appendix A.4).

Neutrinos can also have Majorana masses apart from Dirac mass terms. Since the three right-handed neutrinos are also unified into one  $\underline{\mathbf{3}}_2$  under  $S_4$  the only invariant mass term for the right-handed neutrinos is simply proportional to the unit matrix. The embedding of our model into  $SO(3)_f$  does not change this, however one has to keep in mind that in  $SU(3)_f$  the irreducible three-dimensional representation with which  $\underline{\mathbf{3}}_2$  of  $S_4$  is identified is complex and therefore does not allow an invariant direct mass term for the right-handed neutrinos. Keeping our model as minimal as possible its embedding into  $SO(3)_f$  instead of  $SU(3)_f$  is therefore preferred. Nevertheless, the inclusion of gauge singlets transforming as  $\underline{\mathbf{6}}$  under  $SU(3)_f$  can give masses to the right-handed neutrinos, if the singlets acquire an appropriate VEV. Since the six-dimensional representation of  $SU(3)_f$  contains a total singlet of  $S_4$  (see appendix A.4), this does not necessarily lead to the breaking of  $S_4$  at a high energy scale. Furthermore notice that the situation changes, if the model is embedded into a GUT like  $SO(10)$  at the same time. This will be discussed below.

A non-trivial structure for the right-handed Majorana mass term requires the introduction of gauge singlets which transform non-trivially under  $S_4$ . To implement the canonical type I seesaw [17] the VEVs of such fields ought to be of the order of  $10^{13}$  GeV. In this case, the flavor symmetry  $S_4$  is broken at this high energy scale rather than the electroweak

scale. Hence we discard this possibility. To keep the model as minimal as possible we also do not include  $SU(2)_L$  Higgs triplets which could give rise to a type II seesaw mass for the light neutrinos [18]. In some classes of  $SO(10)$  models this has become quite popular in order to get a relation between the light neutrino mass matrix and the difference of the down and charged lepton mass matrix [19]. Nevertheless fits using type I seesaw in  $SO(10)$  can also be found [20].

To embed the model as a whole into  $SO(10)$ , the Higgs fields also have to be identified with certain  $SO(10)$  representations. In order to get a tree-level coupling they have to be part of either  $\mathbf{10}$ ,  $\mathbf{120}$  or  $\overline{\mathbf{126}}$  under  $SO(10)$ . Since we fixed the Yukawa couplings to be symmetric in flavor space the representation  $\mathbf{120}$  drops out. Furthermore we observe that we need one  $\overline{\mathbf{126}}$  which transforms trivially under  $S_4$  for the right-handed Majorana mass term. We ought to choose its VEVs such that it does not give rise to a mass term for the left-handed neutrinos, since  $SU(2)_L$  Higgs triplets are absent in our low energy model. The minimal choice of fields would be: six Higgs fields transforming as  $(\mathbf{10}, \mathbf{1}_1 + \mathbf{2} + \mathbf{3}_1)$  under  $(SO(10), S_4)$  and one  $\overline{\mathbf{126}} \sim \mathbf{1}_1$  for the mass term of the right-handed neutrinos. The  $\overline{\mathbf{126}}$  should also contribute to the Dirac mass term of the other fermions, since otherwise the masses of the down quarks and charged leptons were the same. In our opinion this is still not enough. Therefore at best one promotes each SM Higgs doublet to one  $\mathbf{10}$  and one  $\overline{\mathbf{126}}$ , such that the  $SO(10)$  model has six ten- and six 126-dimensional Higgs representations. Among the  $\overline{\mathbf{126}}$ s only the one which transforms trivially under  $S_4$  should develop a VEV at high energies and the rest only at the electroweak scale.

To complete the model one needs at least one further Higgs representation, for example a  $\mathbf{210}$ . This representation together with the  $S_4$  invariant 126-dimensional representation should break  $SO(10)$  down to the SM with the Pati-Salam group as intermediate group. However it should not break the flavor group  $S_4$  and hence has to be assigned to  $\mathbf{1}_1$  under  $S_4$ . With all these Higgs fields we believe that it is possible to make a viable high energy completion of our low energy model.

In the numerical examples given below we fit the masses and mixing angles at the scale  $\mu$  which equals the  $W$  boson mass. In order to perform a fit at the GUT scale instead we would have to take into account the renormalization group running of all masses and couplings which will be complicated in a model with such a rich Higgs structure.

Some issues still need to be discussed. Without constructing the  $SO(10)$  invariant Higgs potential it remains the question whether the advocated VEV configuration can be realized. Since our model contains several ten- and 126-dimensional representations and at least one 210-dimensional one, there is a doublet-doublet splitting problem, i.e. one has to ensure that only six of the SM-like Higgs doublets have masses at the electroweak scale while the others acquire masses around the GUT scale. In the same manner one has to solve the well-known doublet-triplet splitting problem. In general separating the electroweak and the GUT scale will be difficult without supersymmetry (SUSY). Furthermore it has to be guaranteed that the gauge couplings unify at all being at the same time still in the perturbative regime at the GUT scale.

Finally, the  $SO(10) \times S_4$  model is embedded into  $SO(10) \times G_f$ . As already mentioned, the three generations of fermions are identified with the fundamental representation of

Particle	SU(3) <sub>c</sub> × SU(2) <sub>L</sub> × U(1) <sub>Y</sub> rep.	S <sub>4</sub> rep.
Quarks $Q$	$(\underline{\mathbf{3}}, \underline{\mathbf{2}}, +\frac{1}{3})$	$\underline{\mathbf{3}}_2$
Antiquarks $u^c$	$(\overline{\mathbf{3}}, \underline{\mathbf{1}}, -\frac{4}{3})$	$\underline{\mathbf{3}}_2$
Antiquarks $d^c$	$(\overline{\mathbf{3}}, \underline{\mathbf{1}}, +\frac{2}{3})$	$\underline{\mathbf{3}}_2$
Leptons $L$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$	$\underline{\mathbf{3}}_2$
Antileptons $e^c$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, +2)$	$\underline{\mathbf{3}}_2$
Right-handed $\nu_s \nu^c$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0)$	$\underline{\mathbf{3}}_2$
Doublet Higgs $\phi_0$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$	$\underline{\mathbf{1}}_1$
Doublet Higgs $(\phi_1, \phi_2)$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$	$\underline{\mathbf{2}}$
Doublet Higgs $(\xi_1, \xi_2, \xi_3)$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$	$\underline{\mathbf{3}}_1$

**Table 2:** The particle content and its symmetry properties under  $S_4$ .

$G_f$ . The Higgs fields contained in the six ten-dimensional representations of  $SO(10)$  are unified into the two representations  $(\underline{\mathbf{10}}, \underline{\mathbf{1}})$  and  $(\underline{\mathbf{10}}, \underline{\mathbf{5}})$  for  $G_f = SO(3)_f$  and into  $(\underline{\mathbf{10}}, \underline{\mathbf{6}})$  for  $G_f = SU(3)_f$ . The Higgs fields in the  $\underline{\mathbf{126}}$ s are treated in a similar way. Therefore the right-handed neutrinos acquire masses by coupling to  $(\underline{\mathbf{126}}, \underline{\mathbf{1}})$  in case of  $G_f$  being  $SO(3)_f$  while they couple to  $(\underline{\mathbf{126}}, \underline{\mathbf{6}})$ , if  $G_f$  is  $SU(3)_f$ . The  $\underline{\mathbf{210}}$  needed for the gauge symmetry breaking transforms trivially under  $G_f$  in the simplest case. Furthermore we need some  $SO(10)$  gauge singlets breaking  $G_f$  in such a way that only  $S_4$  remains. The smallest non-trivial possibility for  $G_f = SO(3)_f$  is  $\underline{\mathbf{9}}$  and for  $G_f = SU(3)_f$   $\underline{\mathbf{6}}$ , since they contain an  $S_4$  singlet (see appendix A.4). But this we will not consider any further in this paper.

To sum up, our model now contains three generations of fermions all transforming as  $\underline{\mathbf{3}}_2$  and six SM-like Higgs fields transforming as  $\underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$  under  $S_4$ . In the next subsection, we show the arising Dirac and Majorana mass matrices.

### 3.2 Fermion masses

The  $S_4$  invariant Yukawa couplings in our model are

$$\begin{aligned}
\mathcal{L}_Y = & \alpha_0^u (Q_1 u^c + Q_2 c^c + Q_3 t^c) \tilde{\phi}_0 + \alpha_1^u (\sqrt{3} (Q_2 c^c - Q_3 t^c) \tilde{\phi}_1 + (-2 Q_1 u^c + Q_2 c^c + Q_3 t^c) \tilde{\phi}_2) \\
& + \alpha_2^u ((Q_2 t^c + Q_3 c^c) \tilde{\xi}_1 + (Q_1 t^c + Q_3 u^c) \tilde{\xi}_2 + (Q_1 c^c + Q_2 u^c) \tilde{\xi}_3) \\
& + \alpha_0^d (Q_1 d^c + Q_2 s^c + Q_3 b^c) \phi_0 + \alpha_1^d (\sqrt{3} (Q_2 s^c - Q_3 b^c) \phi_1 + (-2 Q_1 d^c + Q_2 s^c + Q_3 b^c) \phi_2) \\
& + \alpha_2^d ((Q_2 b^c + Q_3 s^c) \xi_1 + (Q_1 b^c + Q_3 d^c) \xi_2 + (Q_1 s^c + Q_2 d^c) \xi_3) \\
& + \alpha_0^e (L_1 e^c + L_2 \mu^c + L_3 \tau^c) \phi_0 + \alpha_1^e (\sqrt{3} (L_2 \mu^c - L_3 \tau^c) \phi_1 + (-2 L_1 e^c + L_2 \mu^c + L_3 \tau^c) \phi_2) \\
& + \alpha_2^e ((L_2 \tau^c + L_3 \mu^c) \xi_1 + (L_1 \tau^c + L_3 e^c) \xi_2 + (L_1 \mu^c + L_2 e^c) \xi_3) \\
& + \alpha_0^\nu (L_1 \nu_e^c + L_2 \nu_\mu^c + L_3 \nu_\tau^c) \tilde{\phi}_0 + \alpha_1^\nu (\sqrt{3} (L_2 \nu_\mu^c - L_3 \nu_\tau^c) \tilde{\phi}_1 + (-2 L_1 \nu_e^c + L_2 \nu_\mu^c + L_3 \nu_\tau^c) \tilde{\phi}_2) \\
& + \alpha_2^\nu ((L_2 \nu_\tau^c + L_3 \nu_\mu^c) \tilde{\xi}_1 + (L_1 \nu_\tau^c + L_3 \nu_e^c) \tilde{\xi}_2 + (L_1 \nu_\mu^c + L_2 \nu_e^c) \tilde{\xi}_3) \tag{3.1}
\end{aligned}$$

where the fields  $\tilde{\phi}_{0,1,2}$  and  $\tilde{\xi}_{1,2,3}$  are the conjugates of the fields  $\phi_{0,1,2}$  and  $\xi_{1,2,3}$  related by  $\tilde{\phi} = \epsilon \phi^*$  with  $\epsilon$  being the 2-by-2 anti-symmetric matrix in  $SU(2)_L$  space and the star denotes the complex conjugation.



They lead to the following mass matrices for  $i = u, d, e, \nu$ :

$$\mathcal{M}^i = \begin{pmatrix} \alpha_0^i \phi_0 - 2 \alpha_1^i \phi_2 & \alpha_2^i \xi_3 & \alpha_2^i \xi_2 \\ \alpha_2^i \xi_3 & \alpha_0^i \phi_0 + \alpha_1^i (\sqrt{3} \phi_1 + \phi_2) & \alpha_2^i \xi_1 \\ \alpha_2^i \xi_2 & \alpha_2^i \xi_1 & \alpha_0^i \phi_0 + \alpha_1^i (-\sqrt{3} \phi_1 + \phi_2) \end{pmatrix} \quad (3.2)$$

with the Higgs fields being replaced by their VEVs for the down quarks and the charged leptons and by the complex conjugate of their VEVs for the up quarks and the neutrinos. The sum of the VEVs has to be equal the electroweak scale, i.e.  $\sum_i |\text{VEV}_i|^2 \approx (174 \text{ GeV})^2$ . Note that the fields  $\phi_{0,1,2}$  only appear in the diagonal entries. The contribution coming from the Higgs field  $\phi_0$  is proportional to the unit matrix, since  $\phi_0$  transforms trivially under  $S_4$ . The fields  $\phi_1$  and  $\phi_2$  on the other hand are coupled in such a manner that their contribution is traceless. Finally, the fields  $\xi_i$  which form a triplet under  $S_4$  only induce flavor-changing interactions, i.e. their contributions are encoded in the off-diagonal elements of the mass matrix  $\mathcal{M}^i$ . Generally all the parameters in eq. (3.2) can be complex.

In the case of CP-conservation, we arrive at twelve real Yukawa couplings, five real VEVs for the Higgs doublet fields and the right-handed neutrino mass scale  $M_R$  (see below). The sixth VEV is fixed by the electroweak scale. The 18 couplings and VEVs correspond to the twelve masses for the quarks, the charged leptons and the light neutrinos, the three CKM mixing angles and the three leptonic mixing angles whereof two have been measured. With CP-violation - either explicit or spontaneous - the number of parameters is increased such that it exceeds the number of observables. Nevertheless, it is not apparent that the mass structure restricted by the  $S_4$  symmetry allows one to fit all the data. Therefore, we perform a numerical study in the next section.

It is interesting to note that assigning the fermion generations to  $\mathbf{3}_1$  instead of  $\mathbf{3}_2$  leads to exactly the same mass matrices, but does not allow an embedding into  $G_f$  without adding at least two further chiral generations to complete a representation of  $G_f$  (see appendix A.4).

Next we discuss the neutrino sector. Since the right-handed neutrinos transform as  $\mathbf{3}_2$  under  $S_4$ , their mass matrix  $M_{RR}$  is proportional to the unit matrix, i.e.  $M_{RR} = M_R \mathbb{1}$ . This means that the mass matrix for the light neutrinos arising from the type I seesaw has the form:

$$M_\nu = (-) \frac{1}{M_R} \mathcal{M}^\nu \mathcal{M}^{\nu T} = (-) \frac{1}{M_R} (\mathcal{M}^\nu)^2. \quad (3.3)$$

The last step is allowed, since all the Dirac mass matrices are symmetric by construction.

The fact  $M_{RR} \propto \mathbb{1}$  indicates that the seesaw mechanism cannot be the sole origin of the difference between the quark and the lepton mixings in our model. As we will see below, in some cases there exists the possibility to impose an additional symmetry to maintain the diverse mixing patterns.

With regard to leptogenesis,  $M_{RR} \propto \mathbb{1}$  is a viable starting point for the mechanism of resonant leptogenesis [21], since small radiative corrections can generate the small mass splittings which are needed.

Finally, we want to indicate how the mass matrices change in case of a full  $\text{SO}(10)$  model. First the number of Yukawa couplings is reduced from twelve  $\{\alpha_j^i\}$  with  $i = u, d, e, \nu$

and  $j = 0, 1, 2$  to only six  $\{\alpha_{jR}\}$  ( $j = 0, 1, 2$  and  $R = \underline{\mathbf{10}}$  or  $R = \overline{\mathbf{126}}$ ), i.e. three couplings to  $(\underline{\mathbf{10}}, \underline{\mathbf{11}})$ ,  $(\underline{\mathbf{10}}, \underline{\mathbf{2}})$  and  $(\underline{\mathbf{10}}, \underline{\mathbf{31}})$  and three to  $(\overline{\mathbf{126}}, \underline{\mathbf{11}})$ ,  $(\overline{\mathbf{126}}, \underline{\mathbf{2}})$  and  $(\overline{\mathbf{126}}, \underline{\mathbf{31}})$ . At the same time the number of VEVs is in general increased.

The additional embedding of  $S_4$  into a continuous flavor symmetry  $G_f$  reduces the number of Yukawa couplings to its minimum. In case of  $G_f$  being  $SO(3)_f$  the three Yukawa couplings  $\{\alpha_{jR}\}$  for each  $R = \underline{\mathbf{10}}$  or  $R = \overline{\mathbf{126}}$  can be expressed as two independent ones:  $\alpha_{0R} \equiv \alpha_R$  and  $\alpha_{1R} = \alpha_{2R} \equiv \beta_R$ , since the two- and three-dimensional representation under  $S_4$  will be unified into one five-dimensional one. For  $G_f = SU(3)_f$  we are left with only one Yukawa coupling for each  $R = \underline{\mathbf{10}}$  or  $R = \overline{\mathbf{126}}$ , since the six-dimensional representation of  $SU(3)_f$  breaks up into  $\underline{\mathbf{11}} + \underline{\mathbf{2}} + \underline{\mathbf{31}}$  under  $S_4$ .

At the end, taking this setup does not reduce the number of parameters, but they can be eventually correlated such that predictions can be made.

#### 4. Phenomenology of the mass structures

In this section we show that our model allows viable solutions. For this purpose, we introduce our conventions for the mixing matrices and give the experimental data.

##### 4.1 Conventions for the mixing matrices

The Dirac mass matrices arise from the coupling:

$$y_{ij} L_i^T \epsilon \phi L_j^c$$

for the down-type quarks ( $L = Q, L^c = d^c$ ) and charged leptons ( $L = L, L^c = e^c$ ) and for the up-type ones ( $L = Q, L^c = u^c$ ) and the neutrinos ( $L = L, L^c = \nu^c$ ):

$$y_{ij} L_i^T \epsilon \tilde{\phi} L_j^c$$

The mass matrices for the quarks are diagonalized by:

$$U_u^\dagger \mathcal{M}^u \mathcal{M}^{u\dagger} U_u = \text{diag}(m_u^2, m_c^2, m_t^2), \quad U_d^\dagger \mathcal{M}^d \mathcal{M}^{d\dagger} U_d = \text{diag}(m_d^2, m_s^2, m_b^2)$$

where  $U_u$  and  $U_d$  are the unitary matrices transforming the left-handed up and down quarks to their mass eigenstates. The CKM mixing matrix is given by:

$$V_{\text{CKM}} = U_u^T U_d^*$$

The standard parameterization for  $V_{\text{CKM}}$  is [22]:

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where we use the abbreviations  $s_{ij} = \sin(\theta_{ij})$  and  $c_{ij} = \cos(\theta_{ij})$ . The angles are restricted to lie in the first quadrant and  $\delta$  can take any value between 0 and  $2\pi$ .

Similarly, in the leptonic sector the mass matrix for the charged leptons fulfills the relation:

$$U_l^\dagger \mathcal{M}^l \mathcal{M}^{l\dagger} U_l = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$$

with the unitary matrix  $U_l$  transforming the left-handed charged leptons to their mass eigenstates. The light neutrino mass matrix  $M_\nu$  coming from the type I seesaw is a complex symmetric matrix. Hence it can be diagonalized by  $U_\nu$ :

$$U_\nu^\dagger M_\nu U_\nu^* = \text{diag}(m_1, m_2, m_3), \quad U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = \text{diag}(m_1^2, m_2^2, m_3^2)$$

with the mass eigenvalues  $m_i$  being positive definite. The  $U_{\text{MNS}}$  matrix is defined as

$$\nu_{\alpha L} = \sum_{i=1}^3 U_{\text{MNS}}^{\alpha i} \nu_{iL} \quad \text{for } \alpha = e, \mu, \tau \text{ and } i = 1, 2, 3$$

where the  $\nu_{\alpha L}$  denote the flavor and the  $\nu_{iL}$  the mass eigenstates.

Therefore  $U_{\text{MNS}}$  is expressed as

$$U_{\text{MNS}} = U_l^T U_\nu^* .$$

In the case of Majorana neutrinos the  $U_{\text{MNS}}$  matrix can be factorized in a unitary matrix which is parameterized in the same way as the CKM matrix  $V_{\text{CKM}}$  and a diagonal matrix containing the two Majorana phases  $\varphi_1$  and  $\varphi_2$ .

$$U_{\text{MNS}} = \tilde{V}_{\text{CKM}} \cdot \text{diag}(e^{i\varphi_1}, e^{i\varphi_2}, 1) .$$

Both Majorana phases are taken to fulfill  $0 \leq \varphi_1, \varphi_2 \leq \pi$ .

## 4.2 Experimental data

The quark masses at the scale  $\mu \approx M_W$  are given by [23]:

$$\begin{aligned} m_u &= 2.2 \text{ MeV}, & m_c &= 0.81 \text{ GeV}, & m_t &= 170 \text{ GeV}, \\ m_d &= 4.4 \text{ MeV}, & m_s &= 80 \text{ MeV}, & m_b &= 3.1 \text{ GeV}. \end{aligned}$$

The mixing angles and the CP-phase measured in tree-level processes only are [22]:

$$s_{12} = 0.2243 \pm 0.0016, \quad s_{23} = 0.0413 \pm 0.0015, \quad s_{13} = 0.0037 \pm 0.0005, \quad \delta = 1.05 \pm 0.24 .$$

They are almost independent of the scale  $\mu$  at low energies. To quantify the CP-violation one can introduce the Jarlskog invariant  $\mathcal{J}_{\text{CP}}$  [24]:

$$\mathcal{J}_{\text{CP}} = (2.88 \pm 0.33) \times 10^{-5} .$$

The charged lepton masses at  $\mu \approx M_W$  are:

$$m_e = 511 \text{ keV}, \quad m_\mu = 106 \text{ MeV}, \quad m_\tau = 1.78 \text{ GeV} .$$

In the neutrino sector only the two mass squared differences measured in atmospheric and solar neutrino experiments are known [25]:

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = (7.9_{-0.6}^{+0.6}) \times 10^{-5} \text{ eV}^2, \quad |\Delta m_{31}^2| = |m_3^2 - m_1^2| = (2.2_{-0.5}^{+0.7}) \times 10^{-3} \text{ eV}^2 .$$

The leptonic mixing angles are constrained by experiments:

$$s_{13}^2 \leq 0.031, \quad s_{12}^2 = 0.3_{-0.05}^{+0.04}, \quad s_{23}^2 = 0.5_{-0.12}^{+0.14} .$$

All values observed in neutrino oscillations are given at  $2\sigma$  level. The three possible CP-phases  $\delta$ ,  $\varphi_1$  and  $\varphi_2$  in the leptonic sector have not been measured till today.

### 4.3 Numerical examples

In this section, we present two numerical examples which can reconcile the data apart from the fact that some of the quark mixings turn out to be smaller than the central values.

They correspond to perturbations around two different rank one matrices for the quarks and charged leptons:

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad (4.1)$$

i.e. the matrix  $\mathcal{M}_1$  is the democratic mass matrix, which has been discussed several times in the literature [26], and  $\mathcal{M}_2$  only has a non-vanishing 2-3 block. Matrices of the form of  $\mathcal{M}_2$  are often used as approximation for the light neutrino mass matrix to produce maximal atmospheric mixing [27]. Both matrices are able to explain the strong hierarchies observed in the quark and the charged lepton sector. For the neutrinos it is necessary that their mass matrix significantly deviates from the forms of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  to satisfy the restrictions on the lepton mixing matrix. In particular in our two numerical examples the neutrinos dominantly couple to the Higgs fields  $\phi_{1,2}$ , i.e. their mass matrix has large entries on its diagonal.

In the following we show that certain VEV configurations which reflect our flavor symmetry together with some fine-tuning of the Yukawa couplings allow the matrices  $\mathcal{M}_1$  and  $\mathcal{M}_2$  to result from the general form given in eq. (3.2).

First, we consider the matrix  $\mathcal{M}_1$ . In the CP-conserving case, the Higgs potential has a minimum at which the VEVs of the fields  $\xi_i$  are equal, the VEVs of  $\phi_{1,2}$  are zero and  $\phi_0$  has a non-vanishing VEV. The VEVs of  $\xi_i$  and the field  $\phi_0$  are not necessarily equal, but we make the natural assumption that they are almost the same. With this and the constraint that the Yukawa couplings  $\alpha_0^i$  and  $\alpha_2^i$  should be the same, the Dirac mass matrices  $\mathcal{M}^i$  have the democratic form. We then perturb around this known minimum of the Higgs potential and also allow changes in the Yukawa couplings  $\alpha_j^i$  in order to get mass matrices whose masses and mixing parameters agree with the observed ones. For concreteness:

$$\begin{aligned} \alpha_0^u &= 0.651341 - 0.00001 i, & \alpha_1^u &= -0.0058575 + 0.001286 i, & \alpha_2^u &= 0.651341, \\ \alpha_0^d &= 0.011598 + 0.000219 i, & \alpha_1^d &= -0.00299585 - 0.00098905 i, \\ \alpha_2^d &= 0.011708 - 0.000284 i, \\ \alpha_0^e &= 0.0071585 - 2.0 \cdot 10^{-7} i, & \alpha_1^e &= -0.000375685 + 0.00331405 i, & \alpha_2^e &= 0.0065433, \\ \alpha_0^\nu &= 0.15224 + 0.0906 i, & \alpha_1^\nu &= 1.04426 - 0.018245 i, & \alpha_2^\nu &= 0.08364 + 0.04734 i, \\ \langle \phi_0 \rangle &= (87 + 0.01776 i) \text{ GeV}, & \langle \phi_1 \rangle &= (-2.5912 - 14.1982 i) \text{ GeV}, \\ \langle \phi_2 \rangle &= (1.27 - 7.6236 i) \text{ GeV}, & \langle \xi_1 \rangle &= (86.8607 + 1.14585 i) \text{ GeV}, \\ \langle \xi_2 \rangle &= (87 - 0.11162 i) \text{ GeV}, & \langle \xi_3 \rangle &= (87 + 0.88384 i) \text{ GeV}, \\ M_R &= 4.3 \times 10^{13} \text{ GeV}. \end{aligned}$$

The hierarchies among the Yukawa couplings  $\alpha_0^i \sim \alpha_2^i \gg \alpha_1^i$  for  $i = u, d, e$  and  $\alpha_1^\nu \gg \alpha_2^\nu \sim \alpha_0^\nu$  can nicely be explained by an approximate auxiliary  $Z_2$  under which only the

right-handed neutrinos and the Higgs fields  $\phi_1$  and  $\phi_2$  transform:

$$\nu_i^c \rightarrow -\nu_i^c \quad \text{and} \quad \phi_{1,2} \rightarrow -\phi_{1,2} \quad (4.2)$$

whereas the rest remains invariant. The structure of the resulting mass matrices is then democratic for the quarks and the charged leptons while it is dominated by the (2, 2) entry for the light neutrinos. The numerical values given here lead to:

$$\begin{aligned} m_u &= 2.2 \text{ MeV} , \quad m_c = 0.814 \text{ GeV} , \quad m_t = 169.94 \text{ GeV} , \\ m_d &= 4.46 \text{ MeV} , \quad m_s = 81.2 \text{ MeV} , \quad m_b = 3.05 \text{ GeV} , \\ m_e &= 514 \text{ keV} , \quad m_\mu = 105.5 \text{ MeV} , \quad m_\tau = 1.76 \text{ GeV} , \\ \Delta m_{31}^2 &= 2.3 \times 10^{-3} \text{ eV}^2 , \quad \Delta m_{21}^2 = 7.89 \times 10^{-5} \text{ eV}^2 . \end{aligned}$$

The sum of the three light neutrino masses is  $\sum_i m_i = 0.0614 \text{ eV}$ , i.e. the mass spectrum is strongly hierarchical with the smallest mass  $m_1 \approx 0.0038 \text{ eV}$ . This is well below the mass bounds known from cosmology:  $\sum_i m_i < (0.42 \dots 1.8) \text{ eV}$ . This upper bound depends on whether the measurements of the Lyman  $\alpha$  spectrum are included or not [28].

The quark mixing angles turn out to be

$$s_{12} = 0.2238 , \quad s_{13} = 0.003694 , \quad s_{23} = 0.02831 .$$

Unfortunately,  $s_{23}$  is too small. This may be compensated by radiative corrections. The CP-phase  $\delta$  is 1.174 radian and therefore near the upper bound 1.29 radian. The Jarlskog invariant has the value  $\mathcal{J}_{\text{CP}} = 2.1 \times 10^{-5}$ . More interesting to see is that the neutrino mixing angles can be accommodated:

$$s_{12}^2 = 0.3 , \quad s_{23}^2 = 0.49946 ,$$

and we predict  $|U_{\text{MNS}}^{e3}|$  to be 0.06616. This is within the reach of the next generation experiments [29]. Furthermore, the three leptonic CP-phases are:

$$\delta = 0.9983 , \quad \varphi_1 = 1.579 , \quad \varphi_2 = 1.336 \text{ in radian.}$$

Calculating for completeness the magnitudes of  $|m_{ee}|$  and  $m_\beta$  which can be extracted from neutrinoless double beta decay and beta decay, respectively, one finds:

$$|m_{ee}| = \left| \sum_{i=1}^3 (U_{\text{MNS}}^{ei})^2 m_i \right| = 0.0054 \text{ eV} \quad \text{and} \quad m_\beta = \left( \sum_{i=1}^3 |U_{\text{MNS}}^{ei}|^2 m_i^2 \right)^{1/2} = 0.0069 \text{ eV}$$

These values are at least two orders of magnitude below the current experimental bounds which are  $|m_{ee}| \leq 0.9 \text{ eV}$  [30] and  $m_\beta \leq 2.2 \text{ eV}$  [31]. They are also below the limits of the experiments planned for the next years [33, 32]. The main reason for this is the strong hierarchy in the neutrino mass spectrum.

If we take the second mass matrix  $\mathcal{M}_2$  as starting point, we find the following numerical example:

$$\alpha_0^u = 0.56672 - 0.00001 i , \quad \alpha_1^u = 0.2833 + 0.00001 i , \quad \alpha_2^u = 0.85175 + 0.00608 i ,$$

$$\begin{aligned}
\alpha_0^d &= 0.01028 - 3 \cdot 10^{-6} i, & \alpha_1^d &= 0.005145 - 0.000021 i, & \alpha_2^d &= 0.015597 + 0.000789 i, \\
\alpha_0^e &= 0.0059333 - 9 \cdot 10^{-6} i, & \alpha_1^e &= 0.0029676 - 10^{-6} i, & \alpha_2^e &= 0.0088571 + 0.0010664 i, \\
\alpha_0^\nu &= 0.00333 + 0.028 i, & \alpha_1^\nu &= 0.51567 - 0.01616 i, & \alpha_2^\nu &= -0.01572 + 0.89947 i, \\
\langle \phi_0 \rangle &= (99.9974 + 0.0026 i) \text{ GeV}, & \langle \phi_1 \rangle &= (-2.67385 - 6.79332 i) \text{ GeV}, \\
\langle \phi_2 \rangle &= (99.9923 - 0.02867 i) \text{ GeV}, & \langle \xi_1 \rangle &= (99.9907 - 0.266 i) \text{ GeV}, \\
\langle \xi_2 \rangle &= (0.0058 - 0.38774 i) \text{ GeV}, & \langle \xi_3 \rangle &= (0.04332 - 0.14671 i) \text{ GeV}, \\
M_R &= 6.3 \times 10^{13} \text{ GeV}.
\end{aligned}$$

Here, one can clearly see that the Yukawa couplings have to fulfill the relation:  $\alpha_0^i : \alpha_1^i : \alpha_2^i \approx 2 : 1 : 3$  for  $i = u, d, e$  to produce a matrix with a dominant 2-3 block. The resulting masses and mass squared differences are given by:

$$\begin{aligned}
m_u &= 2.4 \text{ MeV}, & m_c &= 0.812 \text{ GeV}, & m_t &= 170.24 \text{ GeV}, \\
m_d &= 4.4 \text{ MeV}, & m_s &= 80 \text{ MeV}, & m_b &= 3.10 \text{ GeV}, \\
m_e &= 512.6 \text{ keV}, & m_\mu &= 106 \text{ MeV}, & m_\tau &= 1.78 \text{ GeV}, \\
\Delta m_{31}^2 &= 2.4 \times 10^{-3} \text{ eV}^2, & \Delta m_{21}^2 &= 7.59 \times 10^{-5} \text{ eV}^2.
\end{aligned}$$

The mass spectrum for the three light neutrinos is normally ordered and degenerate with  $m_1 \approx 0.1682 \text{ eV}$  and  $\sum_i m_i = 0.512 \text{ eV}$  which coincides with the upper bound of the cosmological measurements, if the Lyman  $\alpha$  data is also taken into account. For the quarks all three mixing angles are about 10% too small:

$$s_{12} = 0.2128, \quad s_{13} = 0.0038, \quad s_{23} = 0.0389.$$

Again, this has to be compensated by radiative corrections. The CP-phase  $\delta$  turns out to be the more severe problem, since  $\delta \approx 0.386$  radian, i.e. the CP-violation generated by this setup is more than a factor of two too small and so is the Jarlskog invariant  $\mathcal{J}_{\text{CP}} = 1.15 \times 10^{-5}$ . The two measured mixing angles in the lepton sector can be adjusted to their currently given best-fit values, i.e.  $s_{12}^2 = 0.306$  and  $s_{23}^2 = 0.506$ . The third mixing angle  $\theta_{13}$  and the three leptonic CP-phases turn out to be:

$$s_{13}^2 = 0.0034 \quad (|U_{\text{MNS}}^{e3}| = 0.0584), \quad \delta = 3.032, \quad \varphi_1 = 3.102, \quad \varphi_2 = 3.081,$$

Again, all phases are given in radian.  $|m_{ee}|$  and  $m_\beta$  have almost the same value  $0.168 \text{ eV}$ , since the neutrino masses are nearly degenerate and all phases are approximately  $\pi$ .  $m_\beta$  is near the limit which will be reached by the KATRIN experiment [32]. Due to the degeneracy of the neutrino mass spectrum also  $|m_{ee}|$  [33] and the sum of the neutrino masses [34] can be measured by the up-coming experiments in the next five to ten years.

The matrix structure is very similar to the one of  $\mathcal{M}_2$  for the quarks as well as for the charged leptons while the matrix for the light neutrinos has approximately the form:

$$|M_\nu| \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \tag{4.3}$$

In this case one also finds that the invoked VEV configuration  $\langle \phi_1 \rangle = \langle \xi_2 \rangle = \langle \xi_3 \rangle = 0$  together with the other VEVs being nearly the same is an allowed minimum of the Higgs potential, if CP is conserved.

In the two examples shown here we have taken all parameters to be complex, i.e. we have assumed explicit CP-violation and have not made use of possible field re-definitions to absorb some of the complex phases. We have done so in order to keep as many (free) parameters as possible in the numerical fit procedure. For example, this does not mean that there is no viable solution in the case of spontaneous CP-violation.

## 5. The Higgs potential and its possible minima

Finally, we discuss the  $S_4$  invariant Higgs potential  $V$  and show that there exist two CP-conserving minima from which the VEV configurations assumed above can arise as “perturbations”.

$$\begin{aligned}
 V = & -\mu_1^2(\phi_0^\dagger\phi_0) - \mu_2^2 \sum_{j=1}^2 \phi_j^\dagger\phi_j - \mu_3^2 \sum_{i=1}^3 \xi_i^\dagger\xi_i \tag{5.1} \\
 & + \lambda_0(\phi_0^\dagger\phi_0)^2 + \lambda_1(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2)^2 + \lambda_2(\phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1)^2 \\
 & + \lambda_3 \left[ (\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1)^2 + (\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2)^2 \right] \\
 & + \sigma_1(\phi_0^\dagger\phi_0)(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2) + \left\{ \sigma_2 \left[ (\phi_0^\dagger\phi_1)^2 + (\phi_0^\dagger\phi_2)^2 \right] + \text{h.c.} \right\} \\
 & + \tilde{\sigma}_2 \left[ |\phi_0^\dagger\phi_1|^2 + |\phi_0^\dagger\phi_2|^2 \right] + \left\{ \sigma_3 \left[ (\phi_0^\dagger\phi_1)(\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1) + (\phi_0^\dagger\phi_2)(\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2) \right] + \text{h.c.} \right\} \\
 & + \lambda_1^\xi \left( \sum_{i=1}^3 \xi_i^\dagger\xi_i \right)^2 + \lambda_2^\xi \left[ 3(\xi_2^\dagger\xi_2 - \xi_3^\dagger\xi_3)^2 + (-2\xi_1^\dagger\xi_1 + \xi_2^\dagger\xi_2 + \xi_3^\dagger\xi_3)^2 \right] \\
 & + \lambda_3^\xi \left[ (\xi_2^\dagger\xi_3 + \xi_3^\dagger\xi_2)^2 + (\xi_1^\dagger\xi_3 + \xi_3^\dagger\xi_1)^2 + (\xi_1^\dagger\xi_2 + \xi_2^\dagger\xi_1)^2 \right] \\
 & + \lambda_4^\xi \left[ (\xi_2^\dagger\xi_3 - \xi_3^\dagger\xi_2)^2 + (\xi_1^\dagger\xi_3 - \xi_3^\dagger\xi_1)^2 + (\xi_1^\dagger\xi_2 - \xi_2^\dagger\xi_1)^2 \right] \\
 & + \tau_1(\phi_0^\dagger\phi_0) \left( \sum_{i=1}^3 \xi_i^\dagger\xi_i \right) + \tau_2 \left( \sum_{j=1}^2 \phi_j^\dagger\phi_j \right) \left( \sum_{i=1}^3 \xi_i^\dagger\xi_i \right) \\
 & + \tau_3 \left[ \sqrt{3}(\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1)(\xi_2^\dagger\xi_2 - \xi_3^\dagger\xi_3) + (\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2)(-2\xi_1^\dagger\xi_1 + \xi_2^\dagger\xi_2 + \xi_3^\dagger\xi_3) \right] \\
 & + \left\{ \kappa_1 \left[ 4(\phi_2^\dagger\xi_1)^2 + (\sqrt{3}\phi_1^\dagger\xi_2 + \phi_2^\dagger\xi_2)^2 + (\sqrt{3}\phi_1^\dagger\xi_3 - \phi_2^\dagger\xi_3)^2 \right] + \text{h.c.} \right\} \\
 & + \tilde{\kappa}_1 \left[ 4|\phi_2^\dagger\xi_1|^2 + |\sqrt{3}\phi_1^\dagger\xi_2 + \phi_2^\dagger\xi_2|^2 + |\sqrt{3}\phi_1^\dagger\xi_3 - \phi_2^\dagger\xi_3|^2 \right] \\
 & + \left\{ \kappa_2 \left[ 4(\phi_1^\dagger\xi_1)^2 + (\sqrt{3}\phi_2^\dagger\xi_2 - \phi_1^\dagger\xi_2)^2 + (\sqrt{3}\phi_2^\dagger\xi_3 + \phi_1^\dagger\xi_3)^2 \right] + \text{h.c.} \right\} \\
 & + \tilde{\kappa}_2 \left[ 4|\phi_1^\dagger\xi_1|^2 + |\sqrt{3}\phi_2^\dagger\xi_2 - \phi_1^\dagger\xi_2|^2 + |\sqrt{3}\phi_2^\dagger\xi_3 + \phi_1^\dagger\xi_3|^2 \right] \\
 & + \left\{ \kappa_3 \left[ 2(\phi_2^\dagger\xi_1)(\xi_2^\dagger\xi_3 + \xi_3^\dagger\xi_2) - (\sqrt{3}\phi_1^\dagger\xi_2 + \phi_2^\dagger\xi_2)(\xi_1^\dagger\xi_3 + \xi_3^\dagger\xi_1) + (\sqrt{3}\phi_1^\dagger\xi_3 - \phi_2^\dagger\xi_3)(\xi_1^\dagger\xi_2 + \xi_2^\dagger\xi_1) \right] \right. \\
 & \left. + \text{h.c.} \right\} \\
 & + \left\{ \kappa_4 \left[ 2(\phi_1^\dagger\xi_1)(\xi_3^\dagger\xi_2 - \xi_2^\dagger\xi_3) + (\sqrt{3}\phi_2^\dagger\xi_2 - \phi_1^\dagger\xi_2)(\xi_1^\dagger\xi_3 - \xi_3^\dagger\xi_1) - (\sqrt{3}\phi_2^\dagger\xi_3 + \phi_1^\dagger\xi_3)(\xi_2^\dagger\xi_1 - \xi_1^\dagger\xi_2) \right] \right. \\
 & \left. + \text{h.c.} \right\} \\
 & + \left\{ \kappa_5 \left[ (\phi_0^\dagger\xi_1)^2 + (\phi_0^\dagger\xi_2)^2 + (\phi_0^\dagger\xi_3)^2 \right] + \text{h.c.} \right\} + \tilde{\kappa}_5 \left[ |\phi_0^\dagger\xi_1|^2 + |\phi_0^\dagger\xi_2|^2 + |\phi_0^\dagger\xi_3|^2 \right]
 \end{aligned}$$

$$\begin{aligned}
& + \left\{ \kappa_6 \left[ (\phi_0^\dagger \xi_1)(\xi_2^\dagger \xi_3 + \xi_3^\dagger \xi_2) + (\phi_0^\dagger \xi_2)(\xi_1^\dagger \xi_3 + \xi_3^\dagger \xi_1) + (\phi_0^\dagger \xi_3)(\xi_1^\dagger \xi_2 + \xi_2^\dagger \xi_1) \right] + \text{h.c.} \right\} \\
& + \left\{ \omega_1 \left[ \sqrt{3}(\phi_0^\dagger \phi_1)(\xi_2^\dagger \xi_2 - \xi_3^\dagger \xi_3) + (\phi_0^\dagger \phi_2)(-2\xi_1^\dagger \xi_1 + \xi_2^\dagger \xi_2 + \xi_3^\dagger \xi_3) \right] + \text{h.c.} \right\} \\
& + \left\{ \omega_2 \left[ 2(\phi_0^\dagger \xi_1)(\phi_2^\dagger \xi_1) - (\phi_0^\dagger \xi_2)(\sqrt{3}\phi_1^\dagger \xi_2 + \phi_2^\dagger \xi_2) + (\phi_0^\dagger \xi_3)(\sqrt{3}\phi_1^\dagger \xi_3 - \phi_2^\dagger \xi_3) \right] + \text{h.c.} \right\} \\
& + \left\{ \omega_3 \left[ 2(\xi_1^\dagger \phi_0)(\phi_2^\dagger \xi_1) - (\xi_2^\dagger \phi_0)(\sqrt{3}\phi_1^\dagger \xi_2 + \phi_2^\dagger \xi_2) + (\xi_3^\dagger \phi_0)(\sqrt{3}\phi_1^\dagger \xi_3 - \phi_2^\dagger \xi_3) \right] + \text{h.c.} \right\}
\end{aligned}$$

where parameters with +h.c. in curly brackets are in general complex (for example the last parameter  $\omega_3$ ) and the rest is real. The Higgs potential has 30 parameters in total. 27 of them are quartic couplings out of which 11 are complex. Interesting to notice, the potential is invariant under the following transformation:

$$\phi_1 \rightarrow -\phi_1 \quad \text{and} \quad \xi_2 \leftrightarrow \xi_3 \quad (5.2)$$

and the fields  $\phi_0$ ,  $\phi_2$  and  $\xi_1$  remain unchanged.

We can parameterize all possible real VEVs as:

$$\begin{aligned}
\langle \phi_0 \rangle &= v_0 \quad , \quad \langle \phi_1 \rangle = u \cos(\alpha) \quad , \quad \langle \phi_2 \rangle = u \sin(\alpha) \quad , \\
\langle \xi_1 \rangle &= v \cos(\beta) \quad , \quad \langle \xi_2 \rangle = v \sin(\beta) \cos(\gamma) \quad , \quad \langle \xi_3 \rangle = v \sin(\beta) \sin(\gamma) \quad .
\end{aligned} \quad (5.3)$$

The potential at the minimum has then the following form:

$$\begin{aligned}
V_{\min} &= -\mu_1^2 v_0^2 - \mu_2^2 u^2 - \mu_3^2 v^2 + \lambda_0 v_0^4 + (\lambda_1 + \lambda_3) u^4 \\
&+ \left( \lambda_1^\xi + 4\lambda_3^\xi \sin^2(\beta) \right) v^4 + \lambda_2^\xi \left[ (2 - 3 \sin^2(\beta))^2 + 3 \sin^4(\beta) \cos^2(2\gamma) \right] v^4 \\
&- \lambda_3^\xi (3 + \cos^2(2\gamma)) \sin^4(\beta) v^4 + (\sigma_1 + 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) v_0^2 u^2 + 2\text{Re}(\sigma_3) \sin(3\alpha) v_0 u^3 \\
&+ (\tau_1 + 2\text{Re}(\kappa_5) + \tilde{\kappa}_5) v_0^2 v^2 + (4\text{Re}(\kappa_1 + \kappa_2) + 2(\tilde{\kappa}_1 + \tilde{\kappa}_2) + \tau_2) u^2 v^2 \\
&+ (2\text{Re}(\kappa_1 - \kappa_2) + \tilde{\kappa}_1 - \tilde{\kappa}_2 + \tau_3) [-\cos(2\alpha) (2 - 3 \sin^2(\beta)) \\
&+ \sqrt{3} \sin(2\alpha) \sin^2(\beta) \cos(2\gamma)] u^2 v^2 \\
&+ 3\text{Re}(\kappa_6) \sin(\beta) \sin(2\beta) \sin(2\gamma) v_0 v^3 \\
&+ 2\text{Re}(\omega_2 + \omega_3 - \omega_1) \left[ \sin(\alpha) (2 - 3 \sin^2(\beta)) - \sqrt{3} \cos(\alpha) \sin^2(\beta) \cos(2\gamma) \right] u v_0 v^2
\end{aligned} \quad (5.4)$$

Note that the couplings  $\lambda_2$ ,  $\lambda_4^\xi$ ,  $\kappa_3$ ,  $\kappa_4$  do not appear in  $V_{\min}$ . One can deduce the following VEV conditions:

$$\begin{aligned}
\frac{\partial V_{\min}}{\partial \alpha} &= 2u \left[ v^2 v_0 \left( \cos(\alpha) (2 - 3 \sin^2(\beta)) + \sqrt{3} \sin(\alpha) \sin^2(\beta) \cos(2\gamma) \right) y_1 \right. \\
&+ u v^2 \left( \sin(2\alpha) (2 - 3 \sin^2(\beta)) + \sqrt{3} \cos(2\alpha) \sin^2(\beta) \cos(2\gamma) \right) y_2 \\
&\left. + 3 v_0 u^2 \cos(3\alpha) \text{Re}(\sigma_3) \right] \quad (5.5a)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V_{\min}}{\partial \beta} &= v^2 \left[ -2\sqrt{3} u v_0 \left( \sqrt{3} \sin(\alpha) + \cos(\alpha) \cos(2\gamma) \right) \sin(2\beta) y_1 \right. \\
&+ \sqrt{3} u^2 \left( \sqrt{3} \cos(2\alpha) + \sin(2\alpha) \cos(2\gamma) \right) \sin(2\beta) y_2 \\
&+ 2 v^2 \left( \sin(4\beta) + \sin^2(\beta) \sin(2\beta) \sin^2(2\gamma) \right) y_3 \\
&\left. + \frac{3}{2} v v_0 (3 \sin(3\beta) - \sin(\beta)) \sin(2\gamma) \text{Re}(\kappa_6) \right] \quad (5.5b)
\end{aligned}$$



$$\begin{aligned}
 \frac{\partial V_{\min}}{\partial \gamma} = & 2 v^2 \sin^2(\beta) \left[ 2 \sqrt{3} u v_0 \cos(\alpha) \sin(2 \gamma) y_1 \right. & (5.5c) \\
 & - \sqrt{3} u^2 \sin(2 \alpha) \sin(2 \gamma) y_2 \\
 & + v^2 \sin^2(\beta) \sin(4 \gamma) y_3 \\
 & \left. + 6 v v_0 \cos(\beta) \cos(2 \gamma) \operatorname{Re}(\kappa_6) \right]
 \end{aligned}$$

with  $y_i$  being defined as:

$$y_1 = \operatorname{Re}(\omega_2 + \omega_3 - \omega_1) \tag{5.6a}$$

$$y_2 = 2 \operatorname{Re}(\kappa_1 - \kappa_2) + \tilde{\kappa}_1 - \tilde{\kappa}_2 + \tau_3 \tag{5.6b}$$

$$y_3 = \lambda_3^\xi - 3 \lambda_2^\xi \tag{5.6c}$$

All three equations eq. (5.5) have to be equal zero. A more restrictive requirement is that all the terms should vanish separately. The  $y_i$ ,  $\operatorname{Re}(\kappa_6)$  and  $\operatorname{Re}(\sigma_3)$  are parameters of the Higgs potential and should not be constrained to vanish in order to avoid accidental symmetries arising in the potential. Therefore their coefficients should vanish separately. This poses restrictions on the angles  $\alpha$ ,  $\beta$  and  $\gamma$  as well as on the moduli  $v$ ,  $u$  and  $v_0$ . Obviously, one of the solutions is given by  $u = 0$  ( $\alpha$  is then no longer a variable),  $\beta = \arccos(1/\sqrt{3})$  and  $\gamma = \pi/4$ , i.e. the VEVs of the fields  $\phi_{1,2}$  vanish, the VEVs of the fields  $\xi_i$  are equal  $\frac{v}{\sqrt{3}}$  and  $\phi_0$  has an in general non-vanishing VEV  $v_0$ . Assuming that all quartic couplings are of the same order and all mass parameters have the same order, it is natural that the VEV for the  $\xi_i$  fields is of the order of  $v_0$ . This means only a slight parameter tuning is necessary to achieve the equivalence of these VEVs as is needed for the zeroth order approximation of the fermion masses in our first numerical example. It is noteworthy that the number of (massless) Goldstone bosons is increased by two, if one additionally sets the parameters  $y_i$ ,  $\operatorname{Re}(\sigma_3)$  and  $\operatorname{Re}(\kappa_6)$  to zero, see appendix B.2. I.e. the restrictive requirement that none of them vanishes turns out to be sufficient to avoid further Goldstone bosons. In the numerical study it is pointed out that an auxiliary  $Z_2$  symmetry can explain the required Yukawa couplings. This  $Z_2$ , if also valid in the Higgs sector (and therefore in the whole Lagrangian), restricts the quartic couplings. It enforces  $\sigma_3$ ,  $\kappa_3$ ,  $\kappa_4$ ,  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  to vanish. As far as we can see this does not create an accidental symmetry in the Higgs potential and so does not change the discussion.

A similar analysis can be done for our second numerical example which enforces the VEVs of  $\phi_0$ ,  $\phi_2$  and  $\xi_1$  to be equal and the other VEVs to vanish in the zeroth approximation. This corresponds to  $\alpha = \frac{\pi}{2}$  and  $\beta = 0$  ( $\gamma$  is then irrelevant). One sees that also in this case the coefficients of the parameters  $y_i$ ,  $\operatorname{Re}(\kappa_6)$  and  $\operatorname{Re}(\sigma_3)$  vanish such that the VEV conditions eq. (5.5) can be fulfilled. Note that the values of  $v_0$ ,  $u$  and  $v$  are not constrained to be equal, but again it is plausible that they are nearly the same, if the mass parameters  $\mu_i$  as well as the quartic couplings of the Higgs potential are chosen to be of similar size. Interestingly, setting the parameters  $y_i$ ,  $\operatorname{Re}(\kappa_6)$  and  $\operatorname{Re}(\sigma_3)$  to zero increases the symmetry of the potential also at this minimum and leads to three further Goldstone bosons (see appendix B.3) which is actually one more than in the case above. By inspecting the Higgs potential  $V$  one finds that at least for real VEVs and parameters of the potential requiring

$y_i = 0$ ,  $\text{Re}(\sigma_3) = 0$  and  $\text{Re}(\kappa_6) = 0$  causes an accidental  $\text{SO}(2)_{\text{acc}}$  symmetry under which  $(\phi_1, \phi_2)^T$  forms a doublet and the other fields remain invariant and an accidental  $\text{SO}(3)_{\text{acc}}$  under which  $(\xi_1, \xi_2, \xi_3)^T$  transforms as triplet and the fields  $\phi_0, \phi_1, \phi_2$  trivially. The VEV configuration with vanishing  $\langle \phi_{1,2} \rangle$  then only breaks  $\text{SO}(3)_{\text{acc}}$  and not  $\text{SO}(2)_{\text{acc}}$  and therefore gives rise to two Goldstone bosons whereas the configuration  $\langle \phi_0 \rangle \neq 0$ ,  $\langle \phi_2 \rangle \neq 0$  and  $\langle \xi_1 \rangle \neq 0$  breaks both accidental symmetries resulting in three Goldstone bosons.

In the limiting case that all mass parameters  $\mu_i$  of the potential are equal, these two minima are exactly degenerate (along with many others), since the value of the potential at the minima is  $-\frac{1}{2}(\mu_1^2 v_0^2 + \mu_3^2 v^2)$  and  $-\frac{1}{2}(\mu_1^2 v_0^2 + \mu_2^2 u^2 + \mu_3^2 v^2)$ , respectively.

Further investigation of the minima with CP-violation which lead to realistic masses and mixing parameters is beyond the scope of the paper, but it is plausible that these minima can be formed through small deformations starting with CP-conserving minima, as done here.

We did not perform any checks of the stability of each minimum and the potential as a whole, since the number of parameters ( $\sim 30$ ) makes us confident that there exists at least one point in the parameter space for each minimum where it fulfills together with the potential all the stability criteria. Furthermore we did not address the question whether the minimum is a local or global one, since this might also only depend on an appropriate choice of the parameters.

## 6. Conclusion and outlook

To conclude, we have presented a low energy model based on the SM gauge group augmented with the flavor symmetry  $S_4$ . In contrast to other flavor models we used the requirement to embed our model into a GUT like  $\text{SO}(10)$  and at the same time into a continuous flavor group like  $\text{SO}(3)_f$  or  $\text{SU}(3)_f$  as guideline for the choice of the transformation properties of fermion and Higgs fields under  $S_4$ . The resulting model is minimal in that sense.

Since the structure of the mass matrices is determined by  $S_4$ , it is not obvious whether we can accommodate all observed masses and mixing angles, even though the model contains as many parameters as observables needed to be fixed in the CP-conserving case.

To check this, we explore two cases which are perturbations around two different rank one mass textures for quarks and charged leptons that can be maintained for two choices of ground states of the theory together with some tuning of the Yukawa couplings. The first is the so-called democratic mass matrix and the second one only has a non-vanishing 2-3 block. We give numerical examples for each that are able to fit the known fermion masses and mixing angles in the quark and lepton sector up to rather small deviations. We believe that invoking radiative corrections will lead to full accordance with the experimental data. The difference between the mixings of quarks and leptons crucially depends on the fact that the form of the mass matrix of the light neutrinos differs strongly from the one of the quarks and charged leptons. In our first example an auxiliary  $Z_2$  can help to explain this difference and in the second one the parameters have to be fine-tuned. Taking the auxiliary  $Z_2$  as an exact symmetry of the theory prevents the model from being embedded into  $\text{SO}(10)$ , since

the right-handed neutrinos transform differently from the other fermions. However, one can still promote our model to an  $SU(5)$  GUT. As the Higgs fields  $\phi_{1,2}$  transform under  $Z_2$  whereas  $\xi_{1,2,3}$  remain invariant, the  $Z_2$  is not compatible with any embedding of our  $S_4$  flavor symmetry into  $SO(3)_f$  and  $SU(3)_f$  without adding further fields.

The right-handed neutrinos are degenerate at tree-level and even more their mass matrix is proportional to the unit matrix. Therefore the large leptonic mixing angles have to be encoded in the structure of the Dirac mass matrices for the neutrinos and charged leptons.

The VEV configurations we used in our numerical examples can only be analyzed in the CP-conserving limit, since the Higgs potential turns out to be quite complicated. Nevertheless these are determined by our flavor symmetry. In contrast to this, the values of the Yukawa couplings are not fixed by  $S_4$ . The question why the top quark is 36 times heavier than the bottom quark remains unanswered, but can possibly be explained, if our model is promoted to  $SO(10) \times G_f$ .

Throughout this paper we have not been concerned with the question how to guarantee that the Higgs spectrum just contains one light uncharged (scalar) Higgs inducing only flavor diagonal interactions like the Higgs in the SM while the rest is heavier. Connected to this is the problem of flavor changing neutral currents and lepton flavor violations which usually arise in models with more than one SM-like Higgs doublet. Typically these effects are negligible, if the masses of the flavor changing Higgs fields are above a few TeV. Systematic calculations are difficult, since the Higgs mass spectrum cannot be evaluated in general cases.

Finally, we want to comment on the possibility of supersymmetrizing our model. Introducing supersymmetry apart from its salient feature to solve the hierarchy problem if it is broken at low energies technically leads to a severe simplification of the Higgs potential, since then all the quartic terms are determined by the D-terms. The danger lies in the fact that this generally leads to large accidental global symmetries in the potential which consequently lead to a number of unwanted Goldstone bosons. Two ways of treating this problem can be found in the literature: a.) breaking the discrete and hence also the accidental symmetries by the soft SUSY breaking terms (for example: [35]) or b.) introduce gauge singlets whose couplings are invariant under the discrete symmetry, but break the accidental ones (for example: [36]). Obviously, the whole situation can change in a grand unified model, since then the Higgs doublet fields can belong to various representations of the GUT which have different invariant couplings (see for example [37] for a SUSY  $SO(10)$  model). Since supersymmetric potentials are restricted to be positive by construction, checks of their stability are easier than for non-supersymmetric ones. These issues are currently under study.

## Acknowledgments

The work of R.N.M. was supported by the National Science Foundation grant no. Phy-0354401 and the Alexander von Humboldt Foundation (the Humboldt Research Award).

This work was supported by the ‘‘Sonderforschungsbereich 375 f ur Astro-Teilchenphysik der Deutschen Forschungsgemeinschaft’’.

## A. Group theory of $S_4$

In this appendix we display the representation matrices, Kronecker products and Clebsch Gordan coefficients to calculate all the terms being invariant under the group  $S_4$ .

### A.1 Representation matrices

The representation matrices fulfilling eq. (2.1) can be chosen as:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad \text{for } \underline{\mathbf{2}},$$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{for } \underline{\mathbf{3}}_1$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{for } \underline{\mathbf{3}}_2.$$

These matrices can be found in [14].

### A.2 Kronecker products

The Kronecker products can be calculated from the above given character table [38].

$$\begin{aligned} \underline{\mathbf{1}}_i \times \underline{\mathbf{1}}_j &= \underline{\mathbf{1}}_{(i+j) \bmod 2 + 1} \quad \forall i \text{ and } j \\ \underline{\mathbf{2}} \times \underline{\mathbf{1}}_i &= \underline{\mathbf{2}} \quad \forall i \\ \underline{\mathbf{3}}_i \times \underline{\mathbf{1}}_j &= \underline{\mathbf{3}}_{(i+j) \bmod 2 + 1} \quad \forall i \text{ and } j \end{aligned} \quad \begin{aligned} \underline{\mathbf{3}}_i \times \underline{\mathbf{2}} &= \underline{\mathbf{3}}_1 + \underline{\mathbf{3}}_2 \quad \forall i \\ \underline{\mathbf{3}}_1 \times \underline{\mathbf{3}}_2 &= \underline{\mathbf{1}}_2 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1 + \underline{\mathbf{3}}_2 \end{aligned}$$

$$[\underline{\mathbf{2}} \times \underline{\mathbf{2}}] = \underline{\mathbf{1}}_1 + \underline{\mathbf{2}}, \quad \{\underline{\mathbf{2}} \times \underline{\mathbf{2}}\} = \underline{\mathbf{1}}_2 \quad \text{and} \quad [\underline{\mathbf{3}}_i \times \underline{\mathbf{3}}_i] = \underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1, \quad \{\underline{\mathbf{3}}_i \times \underline{\mathbf{3}}_i\} = \underline{\mathbf{3}}_2 \quad \forall i$$

where we introduced the notation  $[\mu \times \mu]$  for the symmetric and  $\{\mu \times \mu\}$  for the anti-symmetric part of the product  $\mu \times \mu$ .

Note that  $\nu \times \mu = \mu \times \nu$  for all representations  $\mu$  and  $\nu$ .

### A.3 Clebsch Gordan coefficients

The Clebsch Gordan coefficients can be calculated [39] with the given representation matrices for

$$A, A' \sim \underline{\mathbf{1}}_1, \quad B, B' \sim \underline{\mathbf{1}}_2, \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} \sim \underline{\mathbf{3}}_1 \quad \text{and}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix} \sim \underline{\mathbf{3}}_2.$$

Since we choose all the representation matrices to be real, it also holds:

$$A^* \sim \underline{\mathbf{1}}_1, \quad B^* \sim \underline{\mathbf{1}}_2, \quad \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix} \sim \underline{\mathbf{2}}, \quad \begin{pmatrix} b_1^* \\ b_2^* \\ b_3^* \end{pmatrix} \sim \underline{\mathbf{3}}_1 \quad \text{and} \quad \begin{pmatrix} c_1^* \\ c_2^* \\ c_3^* \end{pmatrix} \sim \underline{\mathbf{3}}_2.$$

The Clebsch Gordan coefficients for the one-dimensional representations are trivial:

$$A A' \sim \underline{\mathbf{1}}_1, \quad A B \sim \underline{\mathbf{1}}_2, \quad B A \sim \underline{\mathbf{1}}_2, \quad B B' \sim \underline{\mathbf{1}}_1$$

as well as the products  $\underline{\mathbf{1}}_1 \times \mu$  of any representation  $\mu$  with the total singlet  $\underline{\mathbf{1}}_1$ :

$$\begin{pmatrix} A a_1 \\ A a_2 \end{pmatrix} \sim \underline{\mathbf{2}}, \quad \begin{pmatrix} A b_1 \\ A b_2 \\ A b_3 \end{pmatrix} \sim \underline{\mathbf{3}}_1, \quad \begin{pmatrix} A c_1 \\ A c_2 \\ A c_3 \end{pmatrix} \sim \underline{\mathbf{3}}_2.$$

And here are the ones for  $\underline{\mathbf{1}}_2 \times \mu$  of any representation  $\mu$ :

$$\begin{pmatrix} -B a_2 \\ B a_1 \end{pmatrix} \sim \underline{\mathbf{2}}, \quad \begin{pmatrix} B b_1 \\ B b_2 \\ B b_3 \end{pmatrix} \sim \underline{\mathbf{3}}_2, \quad \begin{pmatrix} B c_1 \\ B c_2 \\ B c_3 \end{pmatrix} \sim \underline{\mathbf{3}}_1.$$

The Clebsch Gordan coefficients for  $\mu \times \mu$  have the form:

<p>for <math>\underline{\mathbf{2}}</math></p> $a_1 a'_1 + a_2 a'_2 \sim \underline{\mathbf{1}}_1$ $-a_1 a'_2 + a_2 a'_1 \sim \underline{\mathbf{1}}_2$ $\begin{pmatrix} a_1 a'_2 + a_2 a'_1 \\ a_1 a'_1 - a_2 a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}$	<p>for <math>\underline{\mathbf{3}}_1</math></p> $\sum_{j=1}^3 b_j b'_j \sim \underline{\mathbf{1}}_1$ $\begin{pmatrix} \frac{1}{\sqrt{2}}(b_2 b'_2 - b_3 b'_3) \\ \frac{1}{\sqrt{6}}(-2b_1 b'_1 + b_2 b'_2 + b_3 b'_3) \end{pmatrix} \sim \underline{\mathbf{2}}$ $\begin{pmatrix} b_2 b'_3 + b_3 b'_2 \\ b_1 b'_3 + b_3 b'_1 \\ b_1 b'_2 + b_2 b'_1 \end{pmatrix} \sim \underline{\mathbf{3}}_1, \quad \begin{pmatrix} b_3 b'_2 - b_2 b'_3 \\ b_1 b'_3 - b_3 b'_1 \\ b_2 b'_1 - b_1 b'_2 \end{pmatrix} \sim \underline{\mathbf{3}}_2$
<p>for <math>\underline{\mathbf{3}}_2</math></p> $\sum_{j=1}^3 c_j c'_j \sim \underline{\mathbf{1}}_1$ $\begin{pmatrix} \frac{1}{\sqrt{2}}(c_2 c'_2 - c_3 c'_3) \\ \frac{1}{\sqrt{6}}(-2c_1 c'_1 + c_2 c'_2 + c_3 c'_3) \end{pmatrix} \sim \underline{\mathbf{2}}$ $\begin{pmatrix} c_2 c'_3 + c_3 c'_2 \\ c_1 c'_3 + c_3 c'_1 \\ c_1 c'_2 + c_2 c'_1 \end{pmatrix} \sim \underline{\mathbf{3}}_1, \quad \begin{pmatrix} c_3 c'_2 - c_2 c'_3 \\ c_1 c'_3 - c_3 c'_1 \\ c_2 c'_1 - c_1 c'_2 \end{pmatrix} \sim \underline{\mathbf{3}}_2.$	

Note here that the parts belonging to the symmetric part of the product  $\mu \times \mu$  are symmetric under the interchange of unprimed and primed whereas the ones belonging to the anti-symmetric part change sign, i.e. are anti-symmetric.

Note also that for our choice of generators the Clebsch Gordan coefficients for  $\underline{\mathbf{3}}_1 \times \underline{\mathbf{3}}_1$  and  $\underline{\mathbf{3}}_2 \times \underline{\mathbf{3}}_2$  turn out to be the same. For the coupling  $\underline{\mathbf{2}} \times \underline{\mathbf{3}}_1$  the Clebsch Gordan coefficients are

$$\begin{aligned} \begin{pmatrix} a_2 b_1 \\ -\frac{1}{2}(\sqrt{3}a_1 b_2 + a_2 b_2) \\ \frac{1}{2}(\sqrt{3}a_1 b_3 - a_2 b_3) \end{pmatrix} &\sim \underline{\mathbf{3}}_1 & \text{and for } \underline{\mathbf{2}} \times \underline{\mathbf{3}}_2 & \begin{pmatrix} a_1 c_1 \\ \frac{1}{2}(\sqrt{3}a_2 c_2 - a_1 c_2) \\ -\frac{1}{2}(\sqrt{3}a_2 c_3 + a_1 c_3) \end{pmatrix} \sim \underline{\mathbf{3}}_1 \\ \begin{pmatrix} a_1 b_1 \\ \frac{1}{2}(\sqrt{3}a_2 b_2 - a_1 b_2) \\ -\frac{1}{2}(\sqrt{3}a_2 b_3 + a_1 b_3) \end{pmatrix} &\sim \underline{\mathbf{3}}_2 & & \begin{pmatrix} a_2 c_1 \\ -\frac{1}{2}(\sqrt{3}a_1 c_2 + a_2 c_2) \\ \frac{1}{2}(\sqrt{3}a_1 c_3 - a_2 c_3) \end{pmatrix} \sim \underline{\mathbf{3}}_2. \end{aligned}$$

And for  $\underline{\mathbf{3}}_1 \times \underline{\mathbf{3}}_2$  one finds the following combinations:

$$\begin{aligned} \sum_{j=1}^3 b_j c_j &\sim \underline{\mathbf{1}}_2 \\ \begin{pmatrix} \frac{1}{\sqrt{6}}(2b_1 c_1 - b_2 c_2 - b_3 c_3) \\ \frac{1}{\sqrt{2}}(b_2 c_2 - b_3 c_3) \end{pmatrix} &\sim \underline{\mathbf{2}} \\ \begin{pmatrix} b_3 c_2 - b_2 c_3 \\ b_1 c_3 - b_3 c_1 \\ b_2 c_1 - b_1 c_2 \end{pmatrix} &\sim \underline{\mathbf{3}}_1, \quad \begin{pmatrix} b_2 c_3 + b_3 c_2 \\ b_1 c_3 + b_3 c_1 \\ b_1 c_2 + b_2 c_1 \end{pmatrix} \sim \underline{\mathbf{3}}_2. \end{aligned}$$

#### A.4 Embeddings of $S_4$ into $SO(3)$ and $SU(3)$

We only display the resolution of the smallest representations of  $SO(3)$  ( $SU(3)$ ) into irreducible ones of  $S_4$ .

$\underline{SO(3)}$	$\rightarrow$	$\underline{S_4}$		$\underline{SU(3)}$	$\rightarrow$	$\underline{S_4}$
$\underline{\mathbf{1}}$	$\rightarrow$	$\underline{\mathbf{1}}_1$		$\underline{\mathbf{1}}$	$\rightarrow$	$\underline{\mathbf{1}}_1$
$\underline{\mathbf{3}}$	$\rightarrow$	$\underline{\mathbf{3}}_2$		$\underline{\mathbf{3}}$	$\rightarrow$	$\underline{\mathbf{3}}_2$
$\underline{\mathbf{5}}$	$\rightarrow$	$\underline{\mathbf{2}} + \underline{\mathbf{3}}_1$		$\underline{\mathbf{6}}$	$\rightarrow$	$\underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1$
$\underline{\mathbf{7}}$	$\rightarrow$	$\underline{\mathbf{1}}_2 + \underline{\mathbf{3}}_1 + \underline{\mathbf{3}}_2$		$\underline{\mathbf{8}}$	$\rightarrow$	$\underline{\mathbf{2}} + \underline{\mathbf{3}}_1 + \underline{\mathbf{3}}_2$
$\underline{\mathbf{9}}$	$\rightarrow$	$\underline{\mathbf{1}}_1 + \underline{\mathbf{2}} + \underline{\mathbf{3}}_1 + \underline{\mathbf{3}}_2$		$\underline{\mathbf{10}}$	$\rightarrow$	$\underline{\mathbf{1}}_2 + \underline{\mathbf{3}}_1 + 2 \underline{\mathbf{3}}_2$

The first table can be found in [14] and the second one can be calculated with the formulae given in [40].

## B. Minimization of the Higgs potential

### B.1 Remaining VEV conditions

The derivatives  $\frac{\partial V_{\min}}{\partial v}$ ,  $\frac{\partial V_{\min}}{\partial u}$  and  $\frac{\partial V_{\min}}{\partial v_0}$  have the following form:

$$\frac{\partial V_{\min}}{\partial v} = 2v \left( -\mu_3^2 + 2(\lambda_1^\xi + 4\lambda_2^\xi)v^2 \right) + 2vv_0^2 (2\text{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1)$$

$$\begin{aligned}
 & +2 v u^2 (4 \operatorname{Re}(\kappa_1 + \kappa_2) + 2(\tilde{\kappa}_1 + \tilde{\kappa}_2) + \tau_2) + 9 v_0 v^2 \operatorname{Re}(\kappa_6) \sin(\beta) \sin(2\beta) \sin(2\gamma) \\
 & +4 u v_0 v \left[ \sin(\alpha) (2 - 3 \sin^2(\beta)) - \sqrt{3} \cos(\alpha) \sin^2(\beta) \cos(2\gamma) \right] y_1 \\
 & +2 u^2 v \left[ -\cos(2\alpha) (2 - 3 \sin^2(\beta)) + \sqrt{3} \sin(2\alpha) \sin^2(\beta) \cos(2\gamma) \right] y_2 \\
 & +4 v^3 \left[ 4 \sin^2(\beta) - (3 + \cos^2(2\gamma)) \sin^4(\beta) \right] y_3 \\
 \frac{\partial V_{\min}}{\partial v_0} & = 2 v_0 (-\mu_1^2 + 2 \lambda_0 v_0^2) + 2 v_0 u^2 (\sigma_1 + 2 \operatorname{Re}(\sigma_2) + \tilde{\sigma}_2) + 2 u^3 \operatorname{Re}(\sigma_3) \sin(3\alpha) \\
 & +2 v_0 v^2 (2 \operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + 3 v^3 \operatorname{Re}(\kappa_6) \sin(\beta) \sin(2\beta) \sin(2\gamma) \\
 & +2 u v^2 \left[ \sin(\alpha) (2 - 3 \sin^2(\beta)) - \sqrt{3} \cos(\alpha) \sin^2(\beta) \cos(2\gamma) \right] y_1 \\
 \frac{\partial V_{\min}}{\partial u} & = 2 u (-\mu_2^2 + 2 (\lambda_1 + \lambda_3) u^2) + 2 u v_0^2 (\sigma_1 + 2 \operatorname{Re}(\sigma_2) + \tilde{\sigma}_2) + 6 v_0 u^2 \operatorname{Re}(\sigma_3) \sin(3\alpha) \\
 & +2 (4 \operatorname{Re}(\kappa_1 + \kappa_2) + 2(\tilde{\kappa}_1 + \tilde{\kappa}_2) + \tau_2) u v^2 \\
 & +2 v_0 v^2 \left[ \sin(\alpha) (2 - 3 \sin^2(\beta)) - \sqrt{3} \cos(\alpha) \sin^2(\beta) \cos(2\gamma) \right] y_1 \\
 & +2 u v^2 \left[ -\cos(2\alpha) (2 - 3 \sin^2(\beta)) + \sqrt{3} \sin(2\alpha) \sin^2(\beta) \cos(2\gamma) \right] y_2
 \end{aligned}$$

In the following sections we present the Higgs mass matrices  $\mathcal{M}^2$  for the two minima around which we have perturbed to find our numerical solutions shown above. We use the following parameterization for the SM-like Higgs doublets  $\phi$ :

$$\phi = \begin{pmatrix} \text{VEV} + \phi^r + i \phi^i \\ \phi^{cr} + i \phi^{ci} \end{pmatrix}.$$

We define  $\mathcal{M}^2$  as:

$$\mathcal{M}^2 = \left. \frac{\partial^2 V}{\partial \phi^x \partial \phi^x} \right|_{\text{all fields} = 0}$$

where  $\phi, \tilde{\phi} \in \{\phi_0, \phi_1, \phi_2, \xi_1, \xi_2, \xi_3\}$  and  $x \in \{r, i, cr, ci\}$ .

We give the mass matrices in the basis  $\{\xi_1^x, \xi_2^x, \xi_3^x, \phi_1^x, \phi_2^x, \phi_0^x\}$  where  $x = r, i, cr, ci$  for the different components of the Higgs doublet fields. For the calculation of the mass matrices we have assumed that all the parameters in the Higgs potential are real such that there is no mixing between the real and the imaginary parts of the components of the Higgs doublet fields.

## B.2 Mass spectrum for the minimum $\langle \xi_i \rangle = \frac{v}{\sqrt{3}}$ , $\langle \phi_{1,2} \rangle = 0$ and $\langle \phi_0 \rangle = v_0$

The non-trivial VEV conditions in this case are:

$$\begin{aligned}
 v (2 \sqrt{3} v v_0 \operatorname{Re}(\kappa_6) + \frac{2}{3} (3 \lambda_1^\xi + 4 \lambda_3^\xi) v^2 - \mu_3^2 + (2 \operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) v_0^2) & = 0 \\
 \frac{2}{\sqrt{3}} \operatorname{Re}(\kappa_6) v^3 + 2 \lambda_0 v_0^3 - v_0 \mu_1^2 + v^2 v_0 (2 \operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) & = 0
 \end{aligned}$$

The resulting mass matrices for the Higgs fields have the following structure:

$$\mathcal{M}^2 = \begin{pmatrix} m_1 & m_2 & m_2 & 0 & -2m_6 & m_3 \\ \cdot & m_1 & m_2 & \sqrt{3}m_6 & m_6 & m_3 \\ \cdot & \cdot & m_1 & -\sqrt{3}m_6 & m_6 & m_3 \\ \cdot & \cdot & \cdot & m_4 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & m_4 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & m_5 \end{pmatrix}$$

The eigenvalues of such a matrix are given by:

$$\frac{1}{2} \left( m_1 + 2m_2 + m_5 \pm \sqrt{(m_1 + 2m_2 - m_5)^2 + 12m_3^2} \right) \\ \frac{1}{2} \left( m_1 - m_2 + m_4 \pm \sqrt{(m_1 - m_2 - m_4)^2 + 24m_6^2} \right) \quad \text{each two times}$$

The corresponding characteristic polynomial is given by:

$$[-6m_6^2 + (m_1 - m_2 - \chi)(m_4 - \chi)]^2 [-3m_3^2 + (m_1 + 2m_2 - \chi)(m_5 - \chi)] = 0$$

For the fields  $\phi^{cr}$  and  $\phi^{ci}$  the variables  $m_i$  with  $i = 1, \dots, 6$  have the following form:

$$m_1 = 2(2v^2 \lambda_1^\xi - \mu_3^2 + v_0^2 \tau_1) \\ m_2 = \frac{4}{3}v(\sqrt{3}v_0 \operatorname{Re}(\kappa_6) + 2\lambda_3^\xi v) \\ m_3 = \frac{2}{3}v(\sqrt{3}v_0(2\operatorname{Re}(\kappa_5) + \tilde{\kappa}_5) + 2\operatorname{Re}(\kappa_6)v) \\ m_4 = 2(-\mu_2^2 + v_0^2 \sigma_1 + \tau_2 v^2) \\ m_5 = 2(2\lambda_0 v_0^2 - \mu_1^2 + \tau_1 v^2) \\ m_6 = -\frac{2}{3}v(2v \operatorname{Re}(\kappa_3) + \sqrt{3}v_0 \operatorname{Re}(\omega_2 + \omega_3))$$

For the uncharged scalar fields  $\phi^r$ :

$$m_1 = 2 \left( \frac{2}{3}v^2(5\lambda_1^\xi + 8\lambda_2^\xi + 4\lambda_3^\xi) - \mu_3^2 + v_0^2(2\operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) \right) \\ m_2 = \frac{4}{3}v(3\sqrt{3}\operatorname{Re}(\kappa_6)v_0 + 2v(\lambda_1^\xi - 2\lambda_2^\xi + 2\lambda_3^\xi)) \\ m_3 = \frac{4}{3}v(3\operatorname{Re}(\kappa_6)v + \sqrt{3}(2\operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1)v_0) \\ m_4 = 2(-\mu_2^2 + (\sigma_1 + 2\operatorname{Re}(\sigma_2) + \tilde{\sigma}_2)v_0^2 + (4\operatorname{Re}(\kappa_1 + \kappa_2) + 2(\tilde{\kappa}_1 + \tilde{\kappa}_2) + \tau_2)v^2) \\ m_5 = 2(6\lambda_0 v_0^2 - \mu_1^2 + (2\operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1)v^2) \\ m_6 = \frac{4}{\sqrt{3}}v_0 v \operatorname{Re}(\omega_1 - \omega_2 - \omega_3)$$

and for the uncharged pseudo-scalars  $\phi^i$ :

$$m_1 = \frac{4}{3}v^2(3\lambda_1^\xi - 4\lambda_4^\xi) - 2\mu_3^2 + 2v_0^2(\tilde{\kappa}_5 + \tau_1 - 2\operatorname{Re}(\kappa_5))$$



$$\begin{aligned}
 m_2 &= \frac{4}{3} v (\sqrt{3} \operatorname{Re}(\kappa_6) v_0 + 2 (\lambda_3^\xi + \lambda_4^\xi) v) \\
 m_3 &= \frac{4}{3} v (2 \sqrt{3} v_0 \operatorname{Re}(\kappa_5) + \operatorname{Re}(\kappa_6) v) \\
 m_4 &= 2 (-\mu_2^2 + (\sigma_1 - 2 \operatorname{Re}(\sigma_2) + \tilde{\sigma}_2) v_0^2 - (4 \operatorname{Re}(\kappa_1 + \kappa_2) - 2 (\tilde{\kappa}_1 + \tilde{\kappa}_2) - \tau_2) v^2) \\
 m_5 &= 2 (2 \lambda_0 v_0^2 - \mu_1^2 + (\tilde{\kappa}_5 + \tau_1 - 2 \operatorname{Re}(\kappa_5)) v^2) \\
 m_6 &= -\frac{4}{3} v (\operatorname{Re}(\kappa_3 - \sqrt{3} \kappa_4) v + \sqrt{3} \operatorname{Re}(\omega_2) v_0)
 \end{aligned}$$

**B.3 Mass spectrum for the minimum**  $\langle \phi_0 \rangle = v_0$ ,  $\langle \phi_2 \rangle = u$ ,  $\langle \xi_1 \rangle = v$  and  $\langle \xi_{2,3} \rangle = \langle \phi_1 \rangle = 0$

The three non-trivial VEV conditions are:

$$\begin{aligned}
 v (2 v^2 (\lambda_1^\xi + 4 \lambda_2^\xi) - \mu_3^2 + v_0^2 (2 \operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + u^2 (4 (2 \operatorname{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2 \tau_3) \\
 + 4 u v_0 \operatorname{Re}(\omega_2 + \omega_3 - \omega_1)) &= 0 \\
 2 u^3 (\lambda_1 + \lambda_3) - 3 u^2 v_0 \operatorname{Re}(\sigma_3) - \mu_2^2 u + v_0^2 u (\sigma_1 + 2 \operatorname{Re}(\sigma_2) + \tilde{\sigma}_2) \\
 + u v^2 (4 (2 \operatorname{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2 \tau_3) + 2 v^2 v_0 \operatorname{Re}(\omega_2 + \omega_3 - \omega_1) &= 0 \\
 2 v_0^3 \lambda_0 - \mu_1^2 v_0 + v_0 u^2 (\sigma_1 + 2 \operatorname{Re}(\sigma_2) + \tilde{\sigma}_2) - u^3 \operatorname{Re}(\sigma_3) + v^2 v_0 (2 \operatorname{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) \\
 + 2 v^2 u \operatorname{Re}(\omega_2 + \omega_3 - \omega_1) &= 0
 \end{aligned}$$

The mass matrices for the Higgs scalars have a block structure:

$$\mathcal{M}^2 = \begin{pmatrix} m_1 & 0 & 0 & 0 & m_8 & m_9 \\ \cdot & m_2 & m_3 & 0 & 0 & 0 \\ \cdot & \cdot & m_2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & m_4 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & m_5 & m_7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & m_6 \end{pmatrix}$$

The corresponding eigenvalues are:

$$m_4, \quad m_2 \pm m_3$$

and the solutions of the characteristic polynomial:

$$\begin{vmatrix} m_1 - \chi & m_8 & m_9 \\ m_8 & m_5 - \chi & m_7 \\ m_9 & m_7 & m_6 - \chi \end{vmatrix} = 0$$

For the fields  $\phi^{cr}$  and  $\phi^{ci}$  the variables  $m_i$  with  $i = 1, \dots, 9$  have the following form:

$$\begin{aligned}
 m_1 &= 2(2v^2(\lambda_1^\xi + 4\lambda_2^\xi) - \mu_3^2 + v_0^2\tau_1 + u^2(\tau_2 + 2\tau_3) - 4uv_0\operatorname{Re}(\omega_1)) \\
 m_2 &= 2(2v^2(\lambda_1^\xi - 2\lambda_2^\xi) - \mu_3^2 + v_0^2\tau_1 + u^2(\tau_2 - \tau_3) + 2uv_0\operatorname{Re}(\omega_1)) \\
 m_3 &= 4v(2u \operatorname{Re}(\kappa_3) + v_0\operatorname{Re}(\kappa_6)) \\
 m_4 &= 2(2u^2(\lambda_1 - \lambda_3) - \mu_2^2 + v_0^2\sigma_1 + v^2(\tau_2 - 2\tau_3) + 2uv_0\operatorname{Re}(\sigma_3))
 \end{aligned}$$

$$\begin{aligned}
m_5 &= 2(2u^2(\lambda_1 + \lambda_3) - \mu_2^2 + v_0^2\sigma_1 + v^2(\tau_2 + 2\tau_3) - 2uv_0\text{Re}(\sigma_3)) \\
m_6 &= 2(2v_0^2\lambda_0 - \mu_1^2 + u^2\sigma_1 + v^2\tau_1) \\
m_7 &= 2(uv_0(2\text{Re}(\sigma_2) + \tilde{\sigma}_2) - u^2\text{Re}(\sigma_3) - 2v^2\text{Re}(\omega_1)) \\
m_8 &= 4v(2u(2\text{Re}(\kappa_1) + \tilde{\kappa}_1) + v_0\text{Re}(\omega_2 + \omega_3)) \\
m_9 &= 2v(v_0(2\text{Re}(\kappa_5) + \tilde{\kappa}_5) + 2u\text{Re}(\omega_2 + \omega_3))
\end{aligned}$$

For the uncharged scalar fields  $\phi^r$ :

$$\begin{aligned}
m_1 &= 2(6v^2(\lambda_1^\xi + 4\lambda_2^\xi) - \mu_3^2 + v_0^2(2\text{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + u^2(4(2\text{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2\tau_3) \\
&\quad + 4uv_0\text{Re}(\omega_2 + \omega_3 - \omega_1)) \\
m_2 &= 2(2v^2(\lambda_1^\xi - 2\lambda_2^\xi + 2\lambda_3^\xi) - \mu_3^2 + v_0^2(2\text{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + u^2(2\text{Re}(\kappa_1) \\
&\quad + \tilde{\kappa}_1 + 3(2\text{Re}(\kappa_2) + \tilde{\kappa}_2) + \tau_2 - \tau_3) + 2uv_0\text{Re}(\omega_1 - \omega_2 - \omega_3)) \\
m_3 &= 12vv_0\text{Re}(\kappa_6) \\
m_4 &= 2(2u^2(\lambda_1 + \lambda_3) - \mu_2^2 + v_0^2(\sigma_1 + 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(4(2\text{Re}(\kappa_2) + \tilde{\kappa}_2) + \tau_2 - 2\tau_3) \\
&\quad + 6uv_0\text{Re}(\sigma_3)) \\
m_5 &= 2(6u^2(\lambda_1 + \lambda_3) - \mu_2^2 + v_0^2(\sigma_1 + 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(4(2\text{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2\tau_3) \\
&\quad - 6uv_0\text{Re}(\sigma_3)) \\
m_6 &= 2(6v_0^2\lambda_0 - \mu_1^2 + u^2(\sigma_1 + 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(2\text{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1)) \\
m_7 &= 2(2v^2\text{Re}(\omega_2 + \omega_3 - \omega_1) + 2uv_0(\sigma_1 + 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) - 3u^2\text{Re}(\sigma_3)) \\
m_8 &= 4v(u(4(2\text{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2\tau_3) + 2v_0\text{Re}(\omega_2 + \omega_3 - \omega_1)) \\
m_9 &= 4v(v_0(2\text{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + 2u\text{Re}(\omega_2 + \omega_3 - \omega_1))
\end{aligned}$$

and for the uncharged pseudo-scalars  $\phi^i$ :

$$\begin{aligned}
m_1 &= 2(2v^2(\lambda_1^\xi + 4\lambda_2^\xi) - \mu_3^2 + v_0^2(\tilde{\kappa}_5 - 2\text{Re}(\kappa_5) + \tau_1) + u^2(4(\tilde{\kappa}_1 - 2\text{Re}(\kappa_1)) + \tau_2 + 2\tau_3) \\
&\quad - 4uv_0\text{Re}(\omega_1 + \omega_2 - \omega_3)) \\
m_2 &= -2(2v^2(-\lambda_1^\xi + 2\lambda_2^\xi + 2\lambda_4^\xi) + \mu_3^2 + v_0^2(2\text{Re}(\kappa_5) - \tilde{\kappa}_5 - \tau_1) + u^2(2\text{Re}(\kappa_1) \\
&\quad - \tilde{\kappa}_1 + 3(2\text{Re}(\kappa_2) - \tilde{\kappa}_2) - \tau_2 + \tau_3) - 2uv_0\text{Re}(\omega_1 + \omega_2 - \omega_3)) \\
m_3 &= 4v(2u\text{Re}(\kappa_3 - \sqrt{3}\kappa_4) + v_0\text{Re}(\kappa_6)) \\
m_4 &= 2(2u^2(\lambda_1 - 2\lambda_2 - \lambda_3) - \mu_2^2 + v_0^2(\sigma_1 - 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(4(\tilde{\kappa}_2 - 2\text{Re}(\kappa_2)) + \tau_2 - 2\tau_3) \\
&\quad + 2uv_0\text{Re}(\sigma_3)) \\
m_5 &= 2(2u^2(\lambda_1 + \lambda_3) - \mu_2^2 + v_0^2(\sigma_1 - 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(4(\tilde{\kappa}_1 - 2\text{Re}(\kappa_1)) + \tau_2 + 2\tau_3) \\
&\quad - 2uv_0\text{Re}(\sigma_3)) \\
m_6 &= 2(2v_0^2\lambda_0 - \mu_1^2 + u^2(\sigma_1 - 2\text{Re}(\sigma_2) + \tilde{\sigma}_2) + v^2(\tilde{\kappa}_5 - 2\text{Re}(\kappa_5) + \tau_1)) \\
m_7 &= -2(u^2\text{Re}(\sigma_3) + 2v^2\text{Re}(\omega_1 + \omega_2 - \omega_3) - 4uv_0\text{Re}(\sigma_2)) \\
m_8 &= 8v(4u\text{Re}(\kappa_1) + v_0\text{Re}(\omega_2)) \\
m_9 &= 8v(v_0\text{Re}(\kappa_5) + u\text{Re}(\omega_2))
\end{aligned}$$

## References

- [1] Z.G. Berezhiani and M.Y. Khlopov, *The theory of broken gauge symmetry of generations*, *Sov. J. Nucl. Phys.* **51** (1990) 739;  
 Z.G. Berezhiani and M.Y. Khlopov, *Physical and astrophysical consequences of a violation of the symmetry of generations*, *Sov. J. Nucl. Phys.* **51** (1990) 935;  
 R. Barbieri, L.J. Hall, S. Raby and A. Romanino, *Unified theories with U(2) flavor symmetry*, *Nucl. Phys.* **B 493** (1997) 3 [[hep-ph/9610449](#)];  
 M.-C. Chen and K.T. Mahanthappa, *From CKM matrix to MNS matrix: a model based on supersymmetric SO(10)  $\times$  U(2)<sub>F</sub> symmetry*, *Phys. Rev.* **D 62** (2000) 113007 [[hep-ph/0005292](#)];  
 S. Raby, *A natural framework for bi-large neutrino mixing*, *Phys. Lett.* **B 561** (2003) 119 [[hep-ph/0302027](#)];  
 M.-C. Chen and K.T. Mahanthappa, *Lepton flavor violating decays, soft leptogenesis and SUSY SO(10)*, *Phys. Rev.* **D 70** (2004) 113013 [[hep-ph/0409096](#)];  
 J. Ferrandis,  *$\alpha \approx \pi/2$  from supersymmetric spontaneous flavor breaking*, *Phys. Rev.* **D 73** (2006) 075004 [[hep-ph/0510051](#)];  
 Z. Berezhiani and A. Rossi, *Predictive grand unified textures for quark and neutrino masses and mixings*, *Nucl. Phys.* **B 594** (2001) 113 [[hep-ph/0003084](#)];  
 G.G. Ross and L. Velasco-Sevilla, *Symmetries and fermion masses*, *Nucl. Phys.* **B 653** (2003) 3 [[hep-ph/0208218](#)];  
 S.F. King and G.G. Ross, *Fermion masses and mixing angles from SU(3) family symmetry and unification*, *Phys. Lett.* **B 574** (2003) 239 [[hep-ph/0307190](#)];  
 G.G. Ross, L. Velasco-Sevilla and O. Vives, *Spontaneous CP-violation and non-abelian family symmetry in SUSY*, *Nucl. Phys.* **B 692** (2004) 50 [[hep-ph/0401064](#)];  
 I. de Medeiros Varzielas and G.G. Ross, *SU(3) family symmetry and neutrino bi-tri-maximal mixing*, *Nucl. Phys.* **B 733** (2006) 31 [[hep-ph/0507176](#)];  
 Z. Berezhiani and F. Nesti, *Supersymmetric SO(10) for fermion masses and mixings: rank-1 structures of flavour*, *JHEP* **03** (2006) 041 [[hep-ph/0510011](#)].
- [2] D.B. Kaplan and M. Schmaltz, *Flavor unification and discrete nonabelian symmetries*, *Phys. Rev.* **D 49** (1994) 3741 [[hep-ph/9311281](#)];  
 D.-G. Lee and R.N. Mohapatra, *An SO(10)  $\times$  S<sub>4</sub> scenario for naturally degenerate neutrinos*, *Phys. Lett.* **B 329** (1994) 463 [[hep-ph/9403201](#)];  
 P.H. Frampton and O.C.W. Kong, *Horizontal symmetry for quark and squark masses in supersymmetric SU(5)*, *Phys. Rev. Lett.* **77** (1996) 1699 [[hep-ph/9603372](#)];  
 K.C. Chou and Y.-L. Wu, *Predictive SUSY SO(10)  $\times$   $\Delta(48) \times$  U(1) model for CP-violation, neutrino oscillation, fermion masses and mixings with small  $\tan \beta$* , [hep-ph/9708201](#);  
 A. Aranda, C.D. Carone and R.F. Lebed, *Maximal neutrino mixing from a minimal flavor symmetry*, *Phys. Rev.* **D 62** (2000) 016009 [[hep-ph/0002044](#)];  
 T. Kobayashi, S. Raby and R.-J. Zhang, *Searching for realistic 4d string models with a pati-Salam symmetry: orbifold grand unified theories from heterotic string compactification on a Z<sub>6</sub> orbifold*, *Nucl. Phys.* **B 704** (2005) 3 [[hep-ph/0409098](#)];  
 R. Dermisek and S. Raby, *Bi-large neutrino mixing and CP-violation in an SO(10) SUSY GUT for fermion masses*, *Phys. Lett.* **B 622** (2005) 327 [[hep-ph/0507045](#)];  
 F. Caravaglios and S. Morisi, *Fermion masses in E<sub>6</sub> grand unification with family permutation symmetries*, [hep-ph/0510321](#);  
 N. Haba and K. Yoshioka, *Discrete flavor symmetry, dynamical mass textures and grand unification*, *Nucl. Phys.* **B 739** (2006) 254 [[hep-ph/0511108](#)].

- [3] For example Y.E. Antebi, Y. Nir and T. Volansky, *Solving flavor puzzles with quiver gauge theories*, *Phys. Rev. D* **73** (2006) 075009 [[hep-ph/0512211](#)];  
T. Yamashita, *Anomalous U(1) GUT*, [hep-ph/0503265](#);  
G.L. Kane, S.F. King, I.N.R. Peddie and L. Velasco-Sevilla, *Study of theory and phenomenology of some classes of family symmetry and unification models*, *JHEP* **08** (2005) 083 [[hep-ph/0504038](#)].
- [4] S. Pakvasa and H. Sugawara, *Discrete symmetry and Cabibbo angle*, *Phys. Lett. B* **73** (1978) 61;  
H. Harari, H. Haut and J. Weyers, *Quark masses and Cabibbo angles*, *Phys. Lett. B* **78** (1978) 459;  
E. Derman, *Flavor unification,  $\tau$  decay and  $b$  decay within the six quark six lepton Weinberg-Salam model*, *Phys. Rev. D* **19** (1979) 317;  
E. Ma, *Two derivable relationships among quark masses and mixing angles*, *Phys. Rev. D* **43** (1991) 2761;  
J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, *The flavor symmetry*, *Prog. Theor. Phys.* **109** (2003) 795 [[hep-ph/0302196](#)], erratum *ibid.* **114** (2005) 287;  
T. Kobayashi, J. Kubo and H. Terao, *Exact  $S_3$  symmetry solving the supersymmetric flavor problem*, *Phys. Lett. B* **568** (2003) 83 [[hep-ph/0303084](#)];  
K.-Y. Choi, Y. Kajiyama, H.M. Lee and J. Kubo, *Double suppression of FCNCs in supersymmetric models*, *Phys. Rev. D* **70** (2004) 055004 [[hep-ph/0402026](#)];  
E. Ma, *Permutation symmetry for neutrino and charged-lepton mass matrices*, *Phys. Rev. D* **61** (2000) 033012 [[hep-ph/9909249](#)];  
S.-L. Chen, M. Frigerio and E. Ma, *Large neutrino mixing and normal mass hierarchy: a discrete understanding*, *Phys. Rev. D* **70** (2004) 073008 [[hep-ph/0404084](#)], erratum *ibid.* **D 70** (2004) 079905;  
W. Grimus and L. Lavoura,  *$S_3 \times Z_2$  model for neutrino mass matrices*, *JHEP* **08** (2005) 013 [[hep-ph/0504153](#)];  
R. Dermisek and S. Raby, *Bi-large neutrino mixing and CP-violation in an SO(10) SUSY GUT for fermion masses*, *Phys. Lett. B* **622** (2005) 327 [[hep-ph/0507045](#)];  
L.J. Hall and H. Murayama, *A geometry of the generations*, *Phys. Rev. Lett.* **75** (1995) 3985 [[hep-ph/9508296](#)];  
F. Caravaglios and S. Morisi, *Fermion masses in  $E_6$  grand unification with family permutation symmetries*, [hep-ph/0510321](#);  
N. Haba and K. Yoshioka, *Discrete flavor symmetry, dynamical mass textures and grand unification*, *Nucl. Phys. B* **739** (2006) 254 [[hep-ph/0511108](#)].
- [5] D. Wyler, *Discrete symmetries in the six quark SU(2)  $\times$  U(1) model*, *Phys. Rev. D* **19** (1979) 3369;  
E. Ma and G. Rajasekaran, *Softly broken  $A_4$  symmetry for nearly degenerate neutrino masses*, *Phys. Rev. D* **64** (2001) 113012 [[hep-ph/0106291](#)];  
K.S. Babu, E. Ma and J.W.F. Valle, *Underlying  $A_4$  symmetry for the neutrino mass matrix and the quark mixing matrix*, *Phys. Lett. B* **552** (2003) 207 [[hep-ph/0206292](#)];  
G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions*, *Nucl. Phys. B* **720** (2005) 64 [[hep-ph/0504165](#)];  
A. Zee, *Obtaining the neutrino mixing matrix with the tetrahedral group*, *Phys. Lett. B* **630** (2005) 58 [[hep-ph/0508278](#)];  
G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing,  $A_4$  and the modular symmetry*, *Nucl. Phys. B* **741** (2006) 215 [[hep-ph/0512103](#)];

- X.G. He, Y.Y. Keum and R.R. Volkas,  *$A_4$  flavour symmetry breaking scheme for understanding quark and neutrino mixing angles*, hep-ph/0601001.
- [6] S. Pakvasa and H. Sugawara, *Mass of the  $T$  quark in  $SU(2) \times U(1)$* , *Phys. Lett.* **B 82** (1979) 105;  
 E. Derman and H.-S. Tsao,  *$SU(2) \times U(1) \times S_n$  flavor dynamics and a bound on the number of flavors*, *Phys. Rev.* **D 20** (1979) 1207;  
 S. Pakvasa, H. Sugawara and Y. Yamanaka, *Permutation symmetries and the fermion mass matrix*, *Phys. Rev.* **D 25** (1982) 1895, erratum *ibid.* **D29** (1984) 2135;  
 R.N. Mohapatra, M.K. Parida and G. Rajasekaran, *High scale mixing unification and large neutrino mixing angles*, *Phys. Rev.* **D 69** (2004) 053007 [hep-ph/0301234];  
 E. Ma, *Neutrino mass matrix from  $S_4$  symmetry*, *Phys. Lett.* **B 632** (2006) 352 [hep-ph/0508231].
- [7] D.-G. Lee and R.N. Mohapatra, *An  $SO(10) \times S_4$  scenario for naturally degenerate neutrinos*, *Phys. Lett.* **B 329** (1994) 463 [hep-ph/9403201].
- [8] W. Grimus and L. Lavoura, *A discrete symmetry group for maximal atmospheric neutrino mixing*, *Phys. Lett.* **B 572** (2003) 189 [hep-ph/0305046];  
 W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura and M. Tanimoto, *Lepton mixing angle  $\theta_{13} = 0$  with a horizontal symmetry  $D_4$* , *JHEP* **07** (2004) 078 [hep-ph/0407112];  
 G. Seidl, *Deconstruction and bilarge neutrino mixing*, hep-ph/0301044;  
 T. Kobayashi, S. Raby and R.-J. Zhang, *Searching for realistic 4d string models with a Pati-Salam symmetry: orbifold grand unified theories from heterotic string compactification on a  $Z_6$  orbifold*, *Nucl. Phys.* **B 704** (2005) 3 [hep-ph/0409098];  
 C.D. Carone and R.F. Lebed, *A hexagonal theory of flavor*, *Phys. Rev.* **D 60** (1999) 096002 [hep-ph/9905275];  
 E. Ma, *Polygonal derivation of the neutrino mass matrix*, hep-ph/0409288;  
 S.-L. Chen and E. Ma, *Discrete symmetry and CP phase of the quark mixing matrix*, *Phys. Lett.* **B 620** (2005) 151 [hep-ph/0505064].
- [9] M. Frigerio, S. Kaneko, E. Ma and M. Tanimoto, *Quaternion family symmetry of quarks and leptons*, *Phys. Rev.* **D 71** (2005) 011901 [hep-ph/0409187];  
 P.H. Frampton and A. Rasin, *NonAbelian discrete symmetries, fermion mass textures and large neutrino mixing*, *Phys. Lett.* **B 478** (2000) 424 [hep-ph/9910522];  
 K.S. Babu and J. Kubo, *Dihedral families of quarks, leptons and Higgses*, *Phys. Rev.* **D 71** (2005) 056006 [hep-ph/0411226];  
 P.H. Frampton and T.W. Kephart, *Simple nonabelian finite flavor groups and fermion masses*, *Int. J. Mod. Phys.* **A 10** (1995) 4689 [hep-ph/9409330].
- [10] A. Aranda, C.D. Carone and R.F. Lebed,  *$U(2)$  flavor physics without  $U(2)$  symmetry*, *Phys. Lett.* **B 474** (2000) 170 [hep-ph/9910392];  
 A. Aranda, C.D. Carone and R.F. Lebed, *Maximal neutrino mixing from a minimal flavor symmetry*, *Phys. Rev.* **D 62** (2000) 016009 [hep-ph/0002044].
- [11] K.C. Chou and Y.-L. Wu, *Predictive SUSY  $SO(10) \times \Delta(48) \times U(1)$  model for CP-violation, neutrino oscillation, fermion masses and mixings with small  $\tan \beta$* , hep-ph/9708201;  
 D.B. Kaplan and M. Schmaltz, *Flavor unification and discrete nonabelian symmetries*, *Phys. Rev.* **D 49** (1994) 3741 [hep-ph/9311281];  
 I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Tri-bimaximal neutrino mixing from discrete subgroups of  $SU(3)$  and  $SO(3)$  family symmetry*, hep-ph/0512313.

- [12] Y.B. Zeldovich, I.Y. Kobzarev and L.B. Okun, *Cosmological consequences of the spontaneous breakdown of discrete symmetry*, *Zh. Eksp. Teor. Fiz.* **67** (1974) 3.
- [13] G. Lazarides and Q. Shafi, *Axion models with no domain wall problem*, *Phys. Lett.* **B 115** (1982) 21.
- [14] J.S. Lomont, *Applications of finite groups*, Acad. Press 1959.
- [15] W.M. Fairbairn, T. Fulton and W.H. Klink, *J. Math. Phys.* **5** (1964) 1038;  
A. Hanany and Y.-H. He, *Non-abelian finite gauge theories*, *JHEP* **02** (1999) 013 [[hep-th/9811183](#)].
- [16] S. Califano, *Vibrational states*, Wiley, London 1976.
- [17] P. Minkowski,  $\mu \rightarrow e\gamma$  at a rate of one out of 1-billion muon decays?, *Phys. Lett.* **B 67** (1977) 421;  
M. Gell-Mann, P. Ramond and R. Slansky, in *Supergravity* P. van Nieuwenhuizen et al. eds., North Holland, Amsterdam, 1980;  
T. Yanagida, in *Proceedings of the workshop on the unified theory and the baryon number in the universe*, O. Sawada and A. Sugamoto eds., KEK, Tsukuba, Japan 1979;  
S.L. Glashow, *The future of elementary particle physics*, in *Proceedings of the 1979 Cargèse Summer Institute on Quarks and Leptons*, M. Lévy et al. eds., Plenum Press, New York 1980;  
R.N. Mohapatra and G. Senjanovic, *Neutrino mass and spontaneous parity nonconservation*, *Phys. Rev. Lett.* **44** (1980) 912.
- [18] G. Lazarides, Q. Shafi and C. Wetterich, *Proton lifetime and fermion masses in an SO(10) model*, *Nucl. Phys.* **B 181** (1981) 287;  
R.N. Mohapatra and G. Senjanović, *Neutrino masses and mixings in gauge models with spontaneous parity violation*, *Phys. Rev.* **D 23** (1981) 165.
- [19] K.S. Babu and R.N. Mohapatra, *Predictive neutrino spectrum in minimal SO(10) grand unification*, *Phys. Rev. Lett.* **70** (1993) 2845 [[hep-ph/9209215](#)];  
B. Brahmachari and R.N. Mohapatra, *Unified explanation of the solar and atmospheric neutrino puzzles in a minimal supersymmetric SO(10) model*, *Phys. Rev.* **D 58** (1998) 015001 [[hep-ph/9710371](#)];  
B. Bajc, G. Senjanovic and F. Vissani,  *$b$ - $\tau$  unification and large atmospheric mixing: a case for non-canonical see-saw*, *Phys. Rev. Lett.* **90** (2003) 051802 [[hep-ph/0210207](#)];  
H.S. Goh, R.N. Mohapatra and S.-P. Ng, *Minimal SUSY SO(10),  $b$   $\tau$  unification and large neutrino mixings*, *Phys. Lett.* **B 570** (2003) 215 [[hep-ph/0303055](#)];  
H.S. Goh, R.N. Mohapatra and S. Nasri, *SO(10) symmetry breaking and type-II seesaw*, *Phys. Rev.* **D 70** (2004) 075022 [[hep-ph/0408139](#)];  
W.-M. Yang and Z.-G. Wang, *Fermion masses and flavor mixing in a supersymmetric SO(10) model*, *Nucl. Phys.* **B 707** (2005) 87 [[hep-ph/0406221](#)].
- [20] K. Matsuda, Y. Koide, T. Fukuyama and H. Nishiura, *How far can the SO(10) two Higgs model describe the observed neutrino masses and mixings?*, *Phys. Rev.* **D 65** (2002) 033008 [[hep-ph/0108202](#)], erratum *ibid.* **D 65** (2002) 079904;  
T. Fukuyama and N. Okada, *Neutrino oscillation data versus minimal supersymmetric SO(10) model*, *JHEP* **11** (2002) 011 [[hep-ph/0205066](#)];  
K.S. Babu and C. Macesanu, *Neutrino masses and mixings in a minimal SO(10) model*, *Phys. Rev.* **D 72** (2005) 115003 [[hep-ph/0505200](#)];  
C.S. Aulakh, *Fermion mass hierarchy in the  $\nu$  MSGUT, I. The real core*, [hep-ph/0602132](#).

- [21] A. Pilaftsis, *CP-violation and baryogenesis due to heavy majorana neutrinos*, *Phys. Rev. D* **56** (1997) 5431 [[hep-ph/9707235](#)];  
 A. Pilaftsis and T.E.J. Underwood, *Resonant leptogenesis*, *Nucl. Phys. B* **692** (2004) 303 [[hep-ph/0309342](#)];  
 T. Hambye, J. March-Russell and S.M. West, *TeV scale resonant leptogenesis from supersymmetry breaking*, *JHEP* **07** (2004) 070 [[hep-ph/0403183](#)].
- [22] PARTICLE DATA GROUP collaboration, S. Eidelman et al., *Review of particle physics*, *Phys. Lett. B* **592** (2004) 1.
- [23] J.F. Donoghue, *The weight for random quark masses*, *Phys. Rev. D* **57** (1998) 5499 [[hep-ph/9712333](#)].
- [24] C. Jarlskog, *Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP-violation*, *Phys. Rev. Lett.* **55** (1985) 1039.
- [25] M. Maltoni, T. Schwetz, M.A. Tortola and J.W.F. Valle, *Status of global fits to neutrino oscillations*, *New J. Phys.* **6** (2004) 122 [[hep-ph/0405172](#)].
- [26] H. Fritzsch and Z.Z. Xing, *Mass and flavor mixing schemes of quarks and leptons*, *Prog. Part. Nucl. Phys.* **45** (2000) 1 [[hep-ph/9912358](#)];  
 P. Kaus and S. Meshkov, *A BCS quark mass matrix*, *Mod. Phys. Lett. A* **3** (1988) 1251, erratum *ibid.* **A 4** (1989) 603;  
 H. Fritzsch and J. Plankl, *Flavor democracy and the lepton-quark hierarchy*, *Phys. Lett. B* **237** (1990) 451;  
 P.M. Fishbane and P. Kaus, *Nearly democratic mass matrices and flavor mixing*, *Phys. Rev. D* **49** (1994) 4780;  
 J.I. Silva-Marcos, *Special symmetric quark mass matrices*, *Phys. Lett. B* **443** (1998) 276 [[hep-ph/9807549](#)];  
 T. Teshima and T. Asai, *Small violation of universal Yukawa coupling and neutrino large mixing*, *Prog. Theor. Phys.* **105** (2001) 763 [[hep-ph/0009181](#)];  
 E.K. Akhmedov, G.C. Branco, F.R. Joaquim and J.I. Silva-Marcos, *Neutrino masses and mixing with seesaw mechanism and universal breaking of extended democracy*, *Phys. Lett. B* **498** (2001) 237 [[hep-ph/0008010](#)];  
 J. Harada, *Phenomenological approach to symmetry breaking pattern of democratic mass matrix*, [hep-ph/0205189](#).
- [27] G. Altarelli and F. Feruglio, *Theoretical models of neutrino masses and mixings*, *Springer Tracts Mod. Phys.* **190** (2003) 169 [[hep-ph/0206077](#)];  
 A. de Gouvea, *Deviation of atmospheric mixing from maximal and structure in the leptonic flavor sector*, *Phys. Rev. D* **69** (2004) 093007 [[hep-ph/0401220](#)].
- [28] S. Hannestad, *Neutrino mass bounds from cosmology*, *Nucl. Phys.* **145** (*Proc. Suppl.*) (2005) 313 [[hep-ph/0412181](#)], and references therein.
- [29] F. Ardellier et al., *Letter of intent for double-CHOOZ: a search for the mixing angle  $\theta_{13}$* , [hep-ex/0405032](#);  
 K. Anderson et al., *White paper report on using nuclear reactors to search for a value of  $\theta_{13}$* , [hep-ex/0402041](#);  
 Y. Itow et al., *The JHF-Kamioka neutrino project*, [hep-ex/0106019](#);  
 NOVA collaboration, D. Ayres et al., *Letter of intent to build an off-axis detector to study  $\nu_{\mu} \rightarrow \nu_e$  oscillations with the NUMI neutrino beam*, [hep-ex/0210005](#);

- P. Huber, J. Kopp, M. Lindner, M. Rolinec and W. Winter, *From double CHOOZ to triple CHOOZ: neutrino physics at the CHOOZ reactor complex*, hep-ph/0601266.
- [30] H.V. Klapdor-Kleingrothaus et al., *Latest results from the heidelberg-moscow double-beta-decay experiment*, *Eur. Phys. J. A* **12** (2001) 147 [hep-ph/0103062]; IGEX collaboration, C.E. Aalseth et al., *The Igex  $^{76}\text{Ge}$  neutrinoless double- $\beta$  decay experiment: prospects for next generation experiments*, *Phys. Rev. D* **65** (2002) 092007 [hep-ex/0202026].
- [31] C. Kraus et al., *Final results from phase II of the Mainz neutrino mass search in tritium beta decay*, *Eur. Phys. J. C* **40** (2005) 447 [hep-ex/0412056]; V.M. Lobashev et al., *Direct search for mass of neutrino and anomaly in the tritium beta-spectrum*, *Phys. Lett. B* **460** (1999) 227.
- [32] KATRIN collaboration, A. Osipowicz et al., *KATRIN: a next generation tritium beta decay experiment with sub-eV sensitivity for the electron neutrino mass*, hep-ex/0109033; KATRIN collaboration, G. Drexlin, *KATRIN: direct measurement of a sub-eV neutrino mass*, *Nucl. Phys.* **145** (Proc. Suppl.) (2005) 263.
- [33] NEMO collaboration, Y. Shitov, *Nemo 3 experiment: preliminary results from 2003 data*, nucl-ex/0405030; CUORE collaboration, S. Capelli et al., *Cuoricino last results and CUORE  $r\bar{\nu}d$* , hep-ex/0505045; R. Ardito et al., *CUORE: a cryogenic underground observatory for rare events*, hep-ex/0501010; MAJORANA collaboration, R. Gaitskell et al., *White paper on the Majorana zero-neutrino double- $\beta$  decay experiment*, nucl-ex/0311013; I. Abt et al., *A new  $^{76}\text{Ge}$  double  $\beta$  decay experiment at lngs*, hep-ex/0404039; M. Danilov et al., *Detection of very small neutrino masses in double- $\beta$  decay using laser tagging*, *Phys. Lett. B* **480** (2000) 12 [hep-ex/0002003]; H. Ejiri et al., *Spectroscopy of double- $\beta$  and inverse- $\beta$  decays from  $^{100}\text{Mo}$  for neutrinos*, *Phys. Rev. Lett.* **85** (2000) 2917 [nucl-ex/9911008]; K. Zuber, *COBRA: double  $\beta$  decay searches using CDTE detectors*, *Phys. Lett. B* **519** (2001) 1 [nucl-ex/0105018]; N. Ishihara, T. Ohama and Y. Yamada, *Nucl. Instrum. Meth.* **A373** (1996) 325; S. Yoshida et al., *Candles project for double  $\beta$  decay of  $^{48}\text{Ca}$* , *Nucl. Phys.* **138** (Proc. Suppl.) (2005) 214; G. Bellini et al., *High sensitivity  $2\beta$  decay study of  $^{116}\text{Cd}$  and  $^{100}\text{Mo}$  with the borexino counting test facility (Cameo project)*, *Eur. Phys. J. C* **19** (2001) 43 [nucl-ex/0007012]; C. Aalseth et al., *Neutrinoless double  $\beta$  decay and direct searches for neutrino mass*, hep-ph/0412300.
- [34] S. Hannestad, *Can cosmology detect hierarchical neutrino masses?*, *Phys. Rev. D* **67** (2003) 085017 [astro-ph/0211106]; J. Lesgourgues, S. Pastor and L. Perotto, *Probing neutrino masses with future galaxy redshift surveys*, *Phys. Rev. D* **70** (2004) 045016 [hep-ph/0403296]; M. Kaplinghat, L. Knox and Y.-S. Song, *Determining neutrino mass from the CMB alone*, *Phys. Rev. Lett.* **91** (2003) 241301 [astro-ph/0303344]; K.N. Abazajian and S. Dodelson, *Neutrino mass and dark energy from weak lensing*, *Phys. Rev. Lett.* **91** (2003) 041301 [astro-ph/0212216].



- [35] K.-Y. Choi, Y. Kajiyama, H.M. Lee and J. Kubo, *Double suppression of FCNCs in supersymmetric models*, *Phys. Rev. D* **70** (2004) 055004 [[hep-ph/0402026](#)].
- [36] T. Kobayashi, J. Kubo and H. Terao, *Exact  $S_3$  symmetry solving the supersymmetric flavor problem*, *Phys. Lett. B* **568** (2003) 83 [[hep-ph/0303084](#)].
- [37] C.S. Aulakh and A. Girdhar, *SO(10) à la Pati-Salam*, *Int. J. Mod. Phys. A* **20** (2005) 865 [[hep-ph/0204097](#)];  
B. Bajc, A. Melfo, G. Senjanovic and F. Vissani, *The minimal supersymmetric grand unified theory, I. Symmetry breaking and the particle spectrum*, *Phys. Rev. D* **70** (2004) 035007 [[hep-ph/0402122](#)];  
T. Fukuyama, A. Ilakovac, T. Kikuchi, S. Meljanac and N. Okada, *SO(10) group theory for the unified model building*, *J. Math. Phys.* **46** (2005) 033505 [[hep-ph/0405300](#)].
- [38] M. Hamermesh, *Group theory and its application to physical problems*, Addison-Wesley, Reading 1962.
- [39] J.F. Cornwell, *Group theory in physics: an introduction*, Academic Press, San Diego 1997.
- [40] P.E. Desmier, R.T. Sharp and J. Patera, *Analytic SU(3) states in a finite subgroup basis*, *J. Math. Phys.* **23** (1982) 1393.